5. The locus of the centers of the conics passing through four real fixed points is itself a conic, whose type depends upon the relative position of the four points. Professor Hughes found that this center-locus is a rectangular hyperbola if the points lie on a circle, a circle if the variable point is at the orthocenter of the three fixed points, a degenerate parabola if three of the points are collinear, a hyperbola if the quadrilateral formed by the points is convex, and an ellipse if the quadrilateral is reëntrant.

6. In her paper Professor Daniells proved that if an entire function $F(z)$ of genus p has a finite number q of imaginary roots, then $F'(z)$ has in addition to the number of roots located by means of Rolle's theorem at most $p + q$ other roots, real or imaginary.

7. Professor Smith reported the results of a statistical study which led him to conclude that the achievement of students in college mathematics bears but little relation to the grades obtained in high school courses in mathematics. The same conclusion holds for the standard psychological tests. However, an entrance examination covering the elements of secondary school mathematics may be used to predict, with a fair degree of success, the mathematical ability of the student. It was also pointed out that the high schools could well afford to introduce in the student's senior year a course primarily designed to test out mathematical tastes and aptitudes.

8. Professor Wilson discussed the periodicity of functions satisfying some of the well-known functional equations. Sufficient conditions for periodicity were found and some of the consequences of periodicity were exhibited. As special cases the periodicity of the continuous functions was discussed.

9. In this paper the history of "The rule of double position" was briefly presented, with extracts from old arithmetics, and its disappearance from modern arithmetics was remarked. An algebraic proof of the rule was given.

10. In his second paper Professor Turner gave a geometric construction for the tangent cone from an external point T to the prolate spheroid whose foci are S and H. He proved that this cone is a quadric cone whose focal lines are TS and TH, that the curve of contact C is a plane curve, and that the cones with vertices S and H, and passing through C are right circular cones. He also showed that if the sum of the focal angles of the tangent cone is a right angle, the locus of its vertex is the director sphere of the spheroid. Other properties were stated without proof.

11. In his paper Professor Woods gave the equation of the line defined in the following way: if from any point P on the circumcircle of a given triangle lines are drawn so as to make a given angle θ with the sides (the angle taken in the same sense in each instance) the three points of intersection so obtained are collinear. The name θ-line was assigned to this line. When θ is a right angle the line is known as the pedal (Wallace or Simson) line of the point P. Several properties of θ-lines were given and each result was specialized for Simson lines.

12. In the first part of his paper Professor Reilly presented a general formula that included the amount and present value of an annuity, the amount and

present value of an annuity due, and the present value of a deferred annuity as special cases. In the latter part he showed that the formulas for constructing tables of the amount and present value of annuities by the continuous process were linear difference-equations of the first order, with constant coefficients, whose solutions gave the usual formulas for the amount and present value of annuities certain.

J. F. REILLY, *Secretary-Treasurer.*

IN THE SURNAMED CHOSEN CHEST.

By DAVID EUGENE SMITH, Columbia University.

III. Numismatica Mathematica.[1]

It might naturally be supposed that few mathematicians would ever have been honored by portrait medals struck in recognition of their services to the world. The issuing of such medals was common in the case of kings and of those who fought their battles, and the church has recognized her prelates and saints in the same manner. In recent years the stage and the societies of the fine arts in general have followed the older custom, and certain scientists of great repute have been similarly honored. These tokens of esteem have usually, however, been bestowed upon men who have had some influence with those in power, either the power of office or that of wealth. Only in the nineteenth century did governments, through learned societies, make a real beginning in honoring the deceased scholars of their countries, and in this way many of the medals of mathematical interest came to be struck.

In this collection there are between 125 and 150 medals of this nature, not counting ancient coins selected to represent the development of the numerals, and counters (jetons) that were used in computation in medieval times. Some of those which represent what might be called an indirect or a casual interest in mathematics, such as medals of Abélard, Pestalozzi, Vittorino da Feltre, and Erasmus are not mentioned in the following list, but those relating to workers in pure mathematics or to those who applied a high order of mathematics to their work in astronomy are included. It is thought that such a list may be of service to those who are interested in the history of mathematics and, to a less degree, of mathematical astronomy. All the medals are originals except in the case of a few electrotypes of specimens in the British Museum, and these cases are specified. In each case the approximate diameter of the medal is given in millimeters. The abbreviations *Ob.* and *Rev.* refer to the obverse (face) and reverse of the medal, and r. and l. are used for right and left respectively. Since the list is not primarily for numismatists, the full description has not generally been given, but enough has been included to allow of easy identification.

[1] This is the third of a series of articles relating to certain historical material in the library of the author. The first dealt with association copies of mathematical books (*1925*, 287–294), and the second with manuscripts of oriental mathematical works (*1925*, 393–397).

1. *Arago*, François (1786–1853). Bronze. Ob., bust by A. Bovy, to r. Rev., "A/Arago/ les auditeurs/ de son cours/d' astronomie. 1843." 56 mm.

2. *Arago*. Bronze. Medallion by David d'Angers. Signed "David 1832." 149 mm.

3. *Arago*. Bronze. Ob., bust by A. Dubois to l. Rev., "Institvt/ de France / Academie / des Sciences / Medaille Arago." 67 mm.

4. *Archimedes* (*c.* 287–212 B. C.). Bronze. Ob., bust by Catenacci, to r. "Archimedes Mathesis ac Mechanices Prodigivm." Rev., "Syracvsis N. A. A.C. CIↃ IↃCCCIX in Vrbis Expvgnatione peremptvs/ A. Arnaud Scvl." 41 mm.

5. *Aristotle* (384–322 B. C.). Bronze. Ob., bust to r. Greek inscription. Rev., pegasus. (Electrotype facsimile from original in the British Museum. Probably 16th century.) 50 mm.

6. *Bacon*, Roger (1214–1294). Bronze. Ob., bust by Gayrard, to r. One of the "series Numismatica," "Durand editit," 1818. 41 mm.

7. *Bailly*, Jean Sylvain (1736–1793). Bronze. Ob., bust by Montagny, to r. One of the "Galerie Metallique des Grands Hommes Français," issued in 1821. 41 mm.

8. *Bailly*. Bronze gilt. Ob., bust to r. "J. Sylvain Bailly né à Paris le XV. Sept. MDCCXXXVI." Rev., "Astronome,/ Autheur de l'Histoire / de l'Astronomie / Membre des trois Academies/ Francaise, des Belle Lettres / et des Sciences / Président de l'Assemble6 / Nationale le 17. Juin / elu/er Maire de Paris / le 15. Juillet 1789 / et Hélas (figure of a headsman's axe) . . . 11 Nov. 1793." 41 mm.

9. *Bailly*. Bronze. Ob. as in No. 8, but under the bust "Offert a la Ville." Rev. as in No. 8 but beginning with the words, "Merite Reconnu / Membre / des Trois Académies," the rest as in No. 8, but without the last line. 41 mm.

10. *Bailly*. Bronze. Ob., bust to l. "J. Sylvain Bailly né à Paris en 7bre. 1736. / Décapité le 12 9bre. 1793." Rev., "Premier Président de/ l'Assemblée Natle. / Ier. Maire de Paris / Il périt / douloureusement / sur un échaffaud / victime / de l'ingratitude / populaire." 32 mm.

11. *Bertrand*, L. F. Joseph (1822–1900). Bronze. Ob., bust by J. C. Chaplain, to l. Rev., "A Joseph Bertrand / Mbre de. l'Academie / Française / Secretaire. Perpetual / de l'Academie. des. sciences / pour. honorer. 50. années / de. Devouement. a. la. Science / et. a. l'Enseignement / ses. Eleves. ses. Admirateurs / et. ses. Amis / Mars / 1844–1894." 67 mm. (One of the finest portrait medals of a mathematician ever issued.)

12. *Brahe*, Tycho (1545–1601). Bronze. Ob., bust by Rogat, to r. One of the same series as No. 6. 41 mm.

13. *Buniakovsky*, Victor Yakovlevich (1804–1889). Bronze. Ob., bust by V. Alexeyev. "Doctor of mathematics of the University of Paris since May 19, 1825" (in Russian). Rev., "In memory of fifty years of service to science, May 19, 1875" (in Russian). 70 mm.

14. *Cardan*, Jerome (1501–1576). Bronze. Ob., bust to r. "Hier. Cardanus. Aetatis. A N. XLVIII." Rev., Landscape and muses. "ONEIPON" (Electrotype facsimile from original in the British Museum. 16th century.) 50 mm.

15. *Cassini*, Giovanni Domenico (1625–1712). Ob., bust by Peuvrier, to r. Rev. "Natus / An. M.DC.XXV. / Perinaldo / in Nicaeae / Massiliensium / Comitatu / Obiit / An. M.DCC.XII." One of the same series as No. 6. Dated 1823. 41 mm.

16. *Cassini*, G. D. Ob., bust by Gatteaux, to r. One of the same series as No. 7.

17. *Cassini*, G. D. Ob., bust to r. "Io. Dom. Cassinvs. Archigym. Bonon. Primar. Astron. et R. Acad." Rev., "Facta. copia. coeli." Dated "Bonon. / M.DC.VC." 60 mm.

18. *Cauchy*, Augustin Louis (1789–1857). Bronze. Medallion by David d'Angers. Head to r. Signed "David 1843." 174 mm.

19. *Cavalieri*, Bonaventura (1598–1647). Bronze. Ob., bust by Nesté, to l. "Bonaventura Cavalerivs." Rev., "Pr. Id. Septemb. / An. MDCCCXLIV / Mediol." 43 mm.

20. *Copernicus*, Nicolaus (1473–1543). Bronze. Ob., bust by Durand, to l. One of the same series as No. 6. Dated 1820. 41 mm.

21. *Copernicus*. White metal. Ob., half figure slightly to r. Legend in Polish. Rev., legend in Polish. Dated 1873. 65 mm.

22. *Copernicus*. Bronze. Ob., bust by Petit, to l. One of the same series as No. 6. Dated 1818. 41 mm.

23. *D'Alembert*, Jean le Rond (1717–1783). Bronze. Ob., bust by Gatteaux, to l. "J. D'Alembert." "F. Gatteaux F. 1785." Rev., laurel wreath inclosing "A l'Immortalité." 59 mm.

24. *D'Alembert*. Bronze. Ob., bust by Depaulis, to l. "Jean le Rond D'Alembert." Rev., "Né / à Paris / en M.DCC.XVII./Mort en M.DCC.LXXXIII." One of the same series as No. 7. Dated 1824. 41 mm.

25. *De Moivre*, Abraham (1667–1754). Bronze. Ob., bust by Dassier, to r. "Abrahamus de Moivre." Rev., "Utriusque / Societatis Regalis / Lond. et Berol. / Sodalis. / M.DCCXLI." 55 mm.

26. *Descartes*, René (1596–1650). Bronze. Ob., bust by Henrionnet, to l. "Renatus Cartesius." Rev. "Natus / An. M.D.XCVI. / Hagae Turonicae / in Gallia / Obiit / Holmiae / An. M.DC.L." One of the same series as No. 6. Dated 1822. 41 mm.

27. *Descartes*. Bronze. Ob., bust by Galle, to r. "Rene Descartes." Rev. "Né à la Haye / en Touraine / en M.D.XCVI./ Mort / en M.DC.L." One of the same series as No. 7. Dated 1819. 41 mm.

28. *Descartes*. Bronze. Ob., bust by Dassier, to r. "Rene Descartes." Rev., recumbent figure, two globes, book, two trumpets. "Philosophe / M. 1650." 27 mm.

29. *Descartes*. Bronze. Ob., bust by Rogat, to r. "René Descartes;" below, "E. Rogat 1846." Rev.,: "Lycée Descartes / Concours Général." Presentation medal to L. J. B. Toussaint, 1848.

30. *Descartes*. Bronze gilt. Ob. same as No. 26. Rev., "Né à la Haye (Indre et Loire) / 1596./ Mort à Stockholm 1650 / Son corps / ramené en France / 1666." 50 mm.

31. *Dürer*, Albrecht (1471–1528). Bronze. Bust to r. "Imago. Alberti. Dvreri. Aetatis. sva. LVI." No rev. This would make the date 1521. The workmanship resembles that of a medal in the South Kensington Museum dated 1528.

32. *Dupin*, Charles, Baron (1784–1873). Ob., bust by A. Bovy, to l. "Baron Charles Dupin de l'Académie des Sciences." Rev., "A Charles Dupin / après 50 années / d' heureuse confraternité / les membres de l' Académie des Sciences / ses collaborateurs / ses amis / —— 1818–1868." 51 mm.

33. *Dupin*, Charles. Bronze, lighter color than No. 29, but ob. from the same die. Rev., "Né à Varzy 1784./ De l' Adad.^{e} des Sc.^{es} 1818./Deputé du Tarn 1827./" . . . "Senateur 1852." 51 mm.

34. *Euler*, Léonard (1707–1783). Silver. Ob., bust by Abramson, to l. Rev., "Radio describit orbem / Natus / MDCCVII." 41 mm.

35. *Fermat*, Pierre (*c.* 1608–1665). Bronze. Ob., bust by Desboeufs, to l. Rev., "Né à Toulouse / en M.D.XCV. / Mort / en M.DC.LXV." One of the same series as No. 7. Dated 1822. The date of his birth (M.D.XCV) is, according to the inscription on his tomb, incorrect. 41 mm.

36. *Fermat*. Bronze. Ob., bust by Dubois, from De Puy Maurin's drawing, to l. Rev. "Académie Roy.^{le} des Sciences Inscrip.^{ons} et Bel.^{es} Let.^{es} de Toulouse / 1822." 35 mm.

37. *Fermat*. Same as No. 36, but in silver. 35 mm.

38. *Fossombroni* Vittorio (1754–1844). Bronze. Ob., bust by "Fabrs d'Udine," to r. "Vittorio Fossombroni Politico e Matematico." Rev., "MDCCCXXXXIIII / Perduta / tanta cagione / I Toscani / La Monumentano / L. Muzzi F." 54 mm.

39. *Galileo Galilei* (1564–1642). Bronze. Ob., bust by Gayrard, to r. "Galilæus Galilæi." Rev., "Natus Pisis / in Italia./ An. M.D.LXIV. / Obiit / An. M.DC.XLII." One of the same series as No. 6. Dated 1818. 41 mm.

40. *Galileo Galilei*. Bronze. Ob., bust by Cinganelli, to l. Rev. "Pisa / che lo vide nascere / ne / celebro / il trecentesimo natalizio / à XVIII Febraio / M.D.CCC.LXIV." 55 mm.

41. *Galileo Galilei*. Bronze. Imperfectly cast piece showing the obverse of No. 40.

42. *Galileo Galilei*. Bronze. Ob., bust by Cerbara, to l. Rev., "Conditori / disciplinae / ad. leges. motvs / et. astron. certo / cognoscendas." 41 mm.

43. *Galileo Galilei*. Silver. Ob. the same as No. 40. Rev., "S.P.Q.P / Academia / pristino decori restitvta / omnibvsqve disciplinis aperta/ III non Decem. MDCCCLIX / Vict. Emm. Sabavd. II P.F.A / Ital. Med. Rege Electo." 55 mm.

44. *Galileo Galilei*. Bronze. Ob., bust to r. "Galilevs. Galilei. Patr. Flor. Mathe.^{m} Cele.^{b}" Rev., two graces with astronomical and geometric models. 88 mm.

45. *Galileo Galilei*. Brass. Ob., bust to r. "Galilevs. de. Galilei. Flor." This legend is engraved. No reverse. 64 mm.

46. *Galileo Galilei*. Bronze. Ob., bust to l. "G. Galilei." Cast bronze piece. No reverse. 66 mm.

47. *Galileo Galilei*. Bronze. Ob., bust to l. "Galilevs. de. Galileis. Florentinvs." Rev., plain, with the single word "Archimedes." 58 mm.

48. *Galileo Galilei*. Bronze. Ob., bust to r. "Galilevs Lyncevs." Reverse, figures (telescope, etc.) "Naturamqve novat." 71 mm.

49. *Galileo Galilei.* Lead, bronzed. Ob., bust slightly to l. "G. Galilei." Rev. "RP." Cast piece. 90 mm.

50. *Gassendi,* Pierre (1592–1655). Bronze. Ob., bust by Vatinelle, to r. Rev., "Né / à Chantersier / près Digne / en M.D.XCVIII. / Mort / en M.DC.LVI." One of the same series as No. 7. Dated 1818. The dates of birth and death are both incorrect. 41 mm.

51. *Gassendi.* Bronze. Ob., bust by I.D., to r. Rev., on a tablet. "Philosophe / M. 1653," with winged figure, globe, etc. The date of death is incorrect. 28 mm.

52. *Gauss,* Carl Friedrich (1777–1855). Bronze. Ob., bust by Brehmer, to r. "Carl Friedrich Gauss / M.DCCLXXVII April XXX / MDCCCLV Februar XXIII." Rev., "Königliche Gesellschaft der Wissenschaften zu Göttingen." Dated 1877. 70 mm.

53. *Gerbert* (Pope Sylvester II) (940–1003). Bronze. Papal medal, 16th (?) century. Ob., bust to r. Rev., two papal keys. Inscription probably "Claves Regni Celorvm," but not very legible. Electrotype from the original in the British Museum. 37 mm.

54. *Grandi,* Guido (1671–1742). Ob., profile by Sulvi, to r. "D. Gvido. Grandvs. Abbas. Camald. Mathem. Pis. Vniv." Rev. "EYPHKA / Inveni / An. MDCCXXXVIII / A. Sulvi. F." Various figures. Electrotype from the original bronze piece in the British Museum. 88 mm.

55. *Halley,* Edmund (1656–1724). Bronze. Ob., bust by Dassier, to r. "Edmundus Halley." Rev. "Astronomus / Regis Magnae / Britanniae. / MDCCXLIV." 55 mm.

56. *Hutton,* Charles (1737–1823). Bronze. Ob., bust by B. Wyon, to l. "Carolus Hutton, LL.D., R.S.S. Æt. LXXXV/ 1821." Rev., various figures; "Fulmina belli / Pondusq. terrae / aestimata." 44 mm.

57. *Huygens,* Christiaan (1629–1695). Bronze. Ob., bust by Henrionnet, to r. "Christianus Hugenius." Rev., "Natus/ Hagae Comitum / in Batavia / An. M.DC.XXIX./ Obiit / An. M.DC.XCV. / " One of the same series as No. 6, 41 mm.

58. *Kepler,* Johann (1571–1630). Bronze. Ob., bust by Caqué, to r. "Joannes Keplerus." Rev., "Natus / An M.D.LXXI,/ Viel in / Regno Wirtembergensi / Obiit / An. M.DC.-XXX." One of the same series as No. 6. Dated 1823. 41 mm.

59. *Lacroix,* Sylvestre François (1765–1843). Bronze. Medallion by David d'Angers. Dated 1841. 169 mm.

60. *Lagrange,* Joseph Louis (1736–1813). Bronze. Ob., bust by Donadio, to l. "Joh. Ludovicus Lagrange." Rev. "Natus / Taurini / in Pedemontis / An. M.DCC.XXXVI. / Obiit / An. M.DCCC.XIII." One of the same series as No. 6. Dated in 1822. 41 mm.

61. *Lagrange.* Bronze. Ob., bust by Donadio, to l. "Joseph Louis Lagrange." Rev., "Né / à Turin / en M.DCC.XXXVI. / Mort / en M.DCCC.XIII." One of the same series as No. 7. Dated 1818. 41 mm.

62. *Lagrange.* Bronze. Medallion by David d'Angers. Bust to l. "Louis de la Grange, David." 152 mm.

63. *Lagrange.* Bronze. Ob., bust by Galeazzi, to l. Aloisivs Lagrange." Rev. "Geometras aevi svi / svperavit / antiqvorvm famam / aeqavit." Although bearing the name Aloisius, the head is that of Joseph Louis. 43 mm.

64. *Lalande.* Joseph Jerome François de (1732–1807). Bronze. Ob., bust by Gatteaux, to l. "Jos. Hier. le François de la Lalande. N. Burgt 1732." Electrotype from the original in the British Museum. 42 mm.

65. *Lalande.* Bronze. Medallion, bust by David d'Angers, to l. "De la Lande / ancien directeur de l'observatoire." 165 mm.

66. *Laplace,* Pierre Simon (1749–1827). Cast iron medallion, no reverse. Ob., bust to l. "Pierre Simon Laplace. Né le 23 Mars 1749." 100 mm.

67. *Laplace.* Bronze. Medallion, bust by David d'Angers, to l. "Laplace." 172 mm.

68. *Le Verrier,* Urbain Jean Joseph (1811–1877). Bronze gilt. Ob., bust by Alphée Dubois, to l. "U. J. J. Le-Verrier de l' Académie des Sciences. 1811–1877." Rev., Helios with four horses; planets with figures and names. Dated, 1884. 68 mm. One of the finest medals ever struck in memory of a mathematician and astronomer.

69. *Leibniz,* Gottfried Wilhelm (1646–1716). Bronze. Ob., bust to r. "Godofr. Wilh. L. B. de Leibniz natvs D.XXI.IVN. / I.IVL / MDCXXXXVI." Rev., Fame placing a wreath on an altar. "Academia Regia Borvss. Scient. Primo Praesidi svo. / MDCCCXXXXVI / D II. IVL." 53 mm.

70. *Leibniz.* Same as No. 69, but in silver.

71. *Leibniz.* Same as 69, but of different color.

72. *Leonardo da Vinci* (1452–1519). Bronze. Ob., bust by Hérard, to l. "Leonardvs-

Vincivs-Florentinvs." Rev. Brush, pen, crown. "Scribit. qvam. svscitat. Artem./ 1665." 54 mm.

73. *Leonardo da Vinci.* Bronze. Ob., bust by Cossa, to l. "Leonardo da Vinci. / L. Cossa. F. 1820." Rev., "Vinci I Vaciti primi / Gallo Monarca gli ultima respiri / cloux le mortali spoglie / A. A. /" An electrotype facsimile. 48 mm.

74. *Lobachevsky,* Nicolai Ivanovitch (1793–1856). Bronze. Ob., bust ¼ to r. Name in Russian letters. Rev. Russian inscription of the Physico-Mathematical Society at the University of Kasan, 1895. 42 mm.

75. *Maurolico,* Francesco (1494–1575). Bronze gilt. Ob., bust by Catenacci, to r. "Franc. Mavrolycvs Archimedes alter." "V. Catenacci scvlp. / L. Taglioni con. Neap." Rev., "Messanae nat. Ann. A.C. CIↃCIↃXCIV Ibiqve obiit CIↃIↃLXXV. / A. Arnaud scvlp." Tablet with further inscription. 41 mm.

76. *Maurolico.* Same as No. 75 but in bronze.

77. *Monge, Gaspard* (1746–1818). Bronze. Ob., bust by Gatteaux, to l. "Gaspard Monge." Rev., "Né /à Beaune en M.DCC.XLVI. / Mort en M.DCCC.XVIII." One of the same series as No. 7. Dated 1822. 41 mm.

78. *Monge.* Bronze. Medallion, bust by David d'Angers, to r. "Monge." "David" 171 mm.

79. *Monge.* Bronze gilt. Ob., bust by Vauthier Galle, to l. "Gaspard Monge." Rev., Né / à Beaune 1746 / Membre de l' Académie /" etc. . . . "Mort 1818." 50 mm.

80. *Neudorffer,* Johann, the elder (1497–1563). Silver. Ob., bust to r. "Ioann Nevdorffer Arith. Æ 57." Rev., "Ditat servata fides." 22 mm.

81. *Neudorffer,* Johann, the younger (1543–1581). Silver. Ob., bust to r. "Ioann Nevdorffer F. Arith. Æ 36." Rev. as in No. 80. 21 mm.

82. *Newton,* Isaac (1642–1727). Bronze. Ob., bust by Croker, to l. "Isaacvs Newtonvs." "I.C." Rev., a seated figure, winged head, holding a tablet with a plan of the solar system. "Felix. Cognoscere. Cavsas / M.DCC.XXVI./" The date of death (1726) is according to the old style, still in use at that time in England.

83. *Newton.* Same as in No. 82, but in silver.

84. *Newton.* Silver. Ob., bust ¾ to r. "Isaacus Newtonius." Rev., wreath surrounding inscription: "Eq. Aur. / Philosophus. / Obiit 31. Mart. / 1727. / Natus Annos / 85." 33 mm.

85. *Newton.* Bronze. Ob., bust by Dassier, to r. "Isaacus Newtonius./" "I. Dassier F./" Rev., his tomb in Westminster Abbey. "Nat. 1642. / M. 1726./" 42 mm.

86. *Newton.* Silver. Ob., bust to r. by Roëttiers. "Isaacvs Newtonivs" "Jac. Roëttiers." Rev. "Quæritur / Huic / Alius / Verg. Æneid: / M.DCC.LXXIV." 54 mm.

87. *Newton.* Bronze. Ob., bust to l. by Petit. "Isaacus Newtonius" "Petit. F." Rev., "Natus / Volstropii / in Anglia / An. M.DC.XLII. / Obiit / An. M.DCCXXVII. /" One of the same series as No. 6. 41 mm.

88. *Newton.* A poor reproduction of No. 85, cast from the original, in bronze.

89. *Newton.* Bronze. Same as No. 84, but in bronze.

90. *Newton.* Bronze. Same as No. 86, but in bronze.

91. *Newton.* Bronze gilt. Same as No. 85, but in bronze gilt.

92. *Newton.* Silver. Same as No. 85, but in silver.

93. *Newton.* Tin. Ob., bust to r. "Sir Isaac Newton." "L.B." "Died 20 March 1727." Rev., "Sir Isaac was born at Woolsthorpe in the Soke of Grantham Decr. 25th. 1642 and educated at the Free Grammar School of King Edward the Sixth. Grantham." 45 mm.

94. *Newton.* Tin. Ob. Bust to r. A calendar medal. "A Calendar 1822. S. Isaac Newton." Rev., calendar. 42 mm.

95. *Newton.* Bronze. A halfpenny struck at the mint in 1793. Ob., bust to l. "Sr. Isaac Newton." Rev. "Half Penny / 1793." Design in the center. 27 mm.

96. *Newton.* Bronze. Similar to No. 95. Date, 1793. 27 mm.

97. *Newton.* Bronze. Similar to No. 96. Date, 1793. 27 mm.

98. *Newton.* Bronze. Similar to No. 97. Date, 1793. 27 mm.

99. *Newton.* Bronze. Similar to No. 98. Date, 1793. 27 mm.

100. *Newton.* Bronze. Similar to No. 98. Date, 1793. 27 mm.

Nos. 95–100 are all of the same design but are not all struck from the same die, as slight differences show. Four have milled edges. Two have inscriptions

on the edges. One reads, "Payable in Hull and in London;" the others, "Payable in London Bristol & Lancaster."

101. *Newton.* Bronze. A farthing struck at the mint in 1793. Ob., bust to l. of the same design as in Nos. 95–100. "Sr. Isaac Newton." Rev., "Farthing / 1793." Design in the center. 21 mm.

102. *Newton.* Bronze. Same as No. 101 but 0.5 mm. larger. They seem to have been struck from the same die, but this is not certain. 21.5 mm.

103. *Newton.* Bronze. A farthing struck in 1793. Ob., bust to l. of the same design as in Nos. 100 and 101, but not from either of the above dies. Legend, "Is. Newton." Rev. "Farthing." Monogram F.H. (T.H.?). "1793."

104. *Newton.* Bronze. A farthing struck in 1793. Ob., bust to l., differing considerably from those in Nos. 101–103. "Isaac Newton." Rev. "Farthing." Britannia seated. "1793." 20.5 mm.

105. *Newton.* Bronze. A farthing struck in 1793. Ob., bust to l. "Ic. Newton." Rev., "Farthing." Britannia seated. "1793." 20.5 mm.

106. *Newton.* Bronze. Similar to No. 105, but not from the same die. 21 mm.

107. *Newton.* Bronze. Similar to Nos. 105 and 106, but not from the same die. 22 mm.

108. *Pascal,* Blaise (1623–1662). Bronze. Ob., bust by Dubois, to r. Rev. "Né / à Clermont / en M.DC.XXIII./ Mort en M.DC.LXII." One of the same series as No. 7. 41 mm.

109. *Pascal.* Bronze gilt. Ob., bust by Dantzell, to r. Rev. "Né/à Clermont / (Auvergne)/ 1623./ Mort/ à Paris/ 1662. 50 mm.

110. *Pascal.* Bronze. Ob., bust by Pingret, to l. "Blasius Pascal." On the edge, "Monachii." Rev., "Natus / Claromontio / Arverniae / An. M.DC.XXIII. / Obiit / An M.DC.-LXII./" One of the same series as No. 6. Dated 1823. 41 mm.

111. *Pascal.* Bronze. Ob., bust by I.D., to l. "Blaise Pascal." Rev. "Philosophe / M. 1662." Winged figure. 28 mm.

112. *Poinsot,* Louis (1777–1859). Bronze. Medallion by David d'Angers, profile to r. Dated 1843. 161 mm.

113. *Poisson,* Simeon Denis (1781–1840). Bronze. Medallion by David d'Angers, profile to l. Dated 1830. 169 mm.

114. *Pythagoras* (*c.* 572–*c.* 501 B.C.). Bronze coin of Samos. Obv., seated figure of Pythagoras. "[P]ythagores Sam Ion." in Greek. Electrotype from the original in the British Museum, and so for Nos. 115–119. 29 mm.

115. *Pythagoras.* Bronze coin of Samos. Obvs., similar to No. 114. "Pythag . . . Sam. Ion." 31 mm.

116. *Pythagoras.* Bronze. Similar to No. 115, but of different coinage. "[. . .] thagores Sa . . . Ion." 31 mm.

117. *Pythagoras.* Bronze. Similar to No. 116, but of different coinage. "Pythagores. Sam. Ion." 30 mm.

118. *Pythagoras.* Bronze. The same in style as No. 117, but smaller. The inscription is illegible. 20 mm.

119. *Pythagoras.* Bronze. The same in style as No. 118, but still smaller. "[. . .] Pythag." 19 mm.

120. *Quetelet,* Lambert Adolphe Jacques (1796–1874). Bronze. Medallion by David d'Angers. Profile to r. 160 mm.

121. *Secchi,* Angelo (1818–1878). Bronze. Ob., reclining figure of Father Tiber; wolf at right nursing Romulus and Remus, by C. Voigt. "Alterivs sic Altera / Poscit opem." Rev. "Angelvs / Secchi / Pont. Acad. Tiberinae / Praeses / A. Ab Acad. Inst. LXV / V.C.MMDC-XXIX." 44 mm.

122. *Steiglehner,* Cölestin (George Christoph) (1738–1819). Bronze. Ob., bust by Losch, to r. "Colest. Steiglehner letzter Furst Abt V. St. Emer. in Regensb." Rev. "Geb. 1738 Gest. 1819 / Dem Vielgeliebten Lehrer / dem Steten Beförderer/ der Wissenschaft / und Kunst / widmeten dies Denkmal / seine Freunde / und Verehrer." 41 mm.

123. *Stevin,* Simon (1548–1620). Bronze. Ob., bust by De Hondt, to r. "Simon Stevin / Inaug. MDCCCXLVI. F. De Hondt." Rev., arms, "S.P.O.B." 48 mm.

124. *Traversa,* Giulio. Bronze. Ob., bust by Restelli, ¾ to r. "Giulio Traversa." Rev., "Al / Geometra / Giulio Traversa / " . . . 19 Giugno 1887." 41 mm.

125. *Thales.* (*c.* 640–*c.* 546 B.C.). Bronze. Renaissance medal. Ob., bust to r. "Thaleuos Miletiou" in Greek letters. Rev., standing figure. Electrotype from the original in the British Museum. 30 mm.

126. *Viviani*, Vincentius (1622–1703). Bronze. Ob., bust to r. "Vinc. Viv. Noviss. Magni Galilæi Discipvlvs æt. LXXIX. Qvi primvs, A. Sal. M.D.CIIIII. Rev. "Ostendit æqvas. et sphæricas svperficies nil recti habentes notis rectangvlis." Electrotype from the original in the British Museum. 90 mm.

127. *Wolf*, Christian (1679–1754). Bronze. Ob., bust by J. Dassier, to r. "Christianus Wolfius." Rev. "Sedes Fructusque Perennis." Seated figure of woman, altar, cornucopia. 42 mm.

128. *Wren*, Christopher (1632–1723). Bronze. Ob., bust by W. Wilson, to r. Rev., St. Paul's cathedral, "Christopher Wren Architect—MDCCX / Si monumentum requiris / circumspice." 58 mm.

There are numerous other medals of mathematical interest in the collection, but only one will be mentioned:

129. *Medal commemorating the Success of the Metric System*, struck in 1874, bronze, from a die engraved by J. C. Chaplain. Ob., four seated figures, one holding the standard meter. "Popvlorvm. Concordiae. Sacrum. / Paris. 1872/ J. C. Chaplain fecit / MDCCCLXXIV." Rev., "Nova Pondervm. ac. Mensvrarvm. Ratio. in. Gallia. Institvta. M. Germ. A. Reip. Conditae III./" surrounded by a further inscription. 100 mm.

VELOCITIES IN THE EINSTEIN THEORY.

By WILLIAM LOWELL PUTNAM.[1]

Before dealing with Einstein's hypothesis let us recall the discovery of Michelson in regard to light. He found that if a flash of light be sent to a mirror and reflected, the time from the starting of the flash to its getting back is always the same, if the distance is the same, no matter in what direction it is sent.

As we know, if a farmer is walking behind his hay rigging and hurries up to the horse's head and then hurries back to his place behind the hayrigging, he has not only gone much further but has taken much more time than if the hayrigging had been standing still.

But though the earth is rushing at times through the ether at a speed of at

[1] William Lowell Putnam was born in Roxbury, Mass., November 22, 1861, and graduated from Harvard in 1882, doing very distinguished work in mathematics. Although he followed the law as a life career, his enthusiasm for mathematics never flagged. For years he was chairman of the committee appointed by the Board of Overseers to visit the mathematical department at Harvard. His sympathetic interest and coöperation in all problems of mathematical teaching were a help and inspiration to those engaged in the work. All of them felt a sense of keen personal loss at his death in July, 1924.

Shortly before his death, he had read, among others, my book on the theory of relativity (*Relativity and Modern Physics*). This seems to have led him to write the present article which was found among his papers by Mrs. Putnam. He had talked over with her his desire to make the theory clearer to the layman in as simple language as possible. His very interesting exposition appears here practically as written by him, save for two explanatory footnotes kindly added by Professor H. P. Manning.

What a pity it is that whereas in England brilliant amateurs like Sir Thomas Heath and Major MacMahon have added very greatly to mathematical scholarship, in America the non-professional student of mathematics is so rare as to be almost unknown.—G. D. Birkhoff.

least 30 miles a second, light takes the same time to go to the mirror and back in the direction in which the earth is travelling as it takes in a direction across this motion.

Michelson's proof of this fact is as convincing as is that which supports our belief in most truths of physics. It is about as convincing as the proof that the earth moves around the sun or that ice is lighter than water.

Let us then assume it to be true and by its aid study Newton's laws.

We begin with bodies moving in a single straight line with uniform velocities, and we assume them to act as if their masses were concentrated at their centers of gravity.

In order to use Michelson's discovery we must assume that one of the bodies, A, emits flashes of light which are reflected to it from other bodies B, C, etc.

Suppose an observer at A draws a chart in which the distance measured along the horizontal axis represents the time in seconds (from some selected epoch O) at which light rays passing to the right leave A. The distance measured along the vertical axis represents the time in seconds at which rays passing to the left reach A. All these times can be taken by A.

Suppose now that A has chosen for his epoch, noon, January 1, 1924, and that exactly at the epoch he sends a flash of light to the right towards B, C, and suppose he gets the reflection back from B at two seconds past twelve and from C at five seconds past, and let the sending of the reflections from B, C be called events which we will designate on our time chart as P_1, P_2 respectively. The coördinates of P_1 are (0, 2), of P_2, (0, 5).

In general every event P on any body Q may be designated by two coördinates on A's chart: t_1, the time at which a ray of light going to the right would have to leave A in order to reach Q at the time of the event and t_2 the time at which the ray of light leaving Q at the time of the event P and going to the left reaches A.

Let us consider a series of events for which the coördinates are equal, (0, 0), (1, 1), (2, 2), (3, 3). Clearly they lie on a straight line making angles of 45° with the axes.

For all events on this line the times of sending the original flash and the times of receiving the return flash are the same. These events must all occur on A itself. It is therefore the life line of A, or more briefly A's line. The events (1, 1), (2, 2), (3, 3), etc. occur respectively 1 second, 2 seconds, and 3 seconds later than the epoch but they all occur on A.

In general, the interval $t_2 - t_1$ between the time of sending a flash to the right from A to reach an event P, and the time of receiving the return flash to the left is proportional to the distance from A to P.

If a series of flashes are sent to the right from A to B, and the reflections always return after the same interval, then the distance from A to B is not changing.

Suppose flashes sent to B at the epoch and at intervals of a second afterward are reflected so as to reach A respectively at one second after the epoch, at $2\frac{1}{2}$ seconds after, at 4 seconds after, and at $5\frac{1}{2}$ seconds after. Then the reply to

the one sent at the epoch was received 1 second after the flash was sent, the reply to the one sent 3 seconds after the epoch was received 2½ seconds after the flash was sent. This shows that the bodies are getting uniformly further from each other.

If the intervals between the sending of the flash and the receipt of the reply are growing longer, then the bodies are getting further apart. If the intervals are growing shorter, then the bodies are getting nearer to each other.

As all the bodies we are considering are moving in one straight line in space, each with a uniform velocity, they must either be moving at the same speed and in the same direction as A or they must be approaching A at a uniform rate or receding from A at a uniform rate.

Therefore since A's line on his chart is a straight line the lines of all other bodies as shown on A's chart must also be straight lines. Assume two particles A and B starting from the same epoch and moving each with uniform velocity to the right, B moving faster than A. These are shown in two positions in Fig. 1.

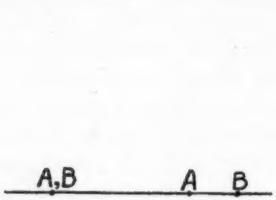

FIG. 1.

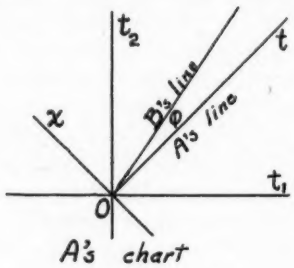

FIG. 2.

In the second position B is some distance to the right of A and the speed with which this distance increases, that is, the distance divided by the time from the epoch is the velocity of B with respect to A or, in a reverse direction, of A with respect to B.

Evidently the speed with which B is receding from A is proportional to the tangent of the angle between the life lines of A and B on A's chart (Fig. 2).[1] If φ represent this angle and v this velocity then $v = k \tan \varphi$ where the value of the constant k depends on our unit of velocity.

The vertical axis of A's chart represents a ray of light travelling to the right from the epoch. Let us assume the velocity of light as our unit; hence for this particular line $v = 1$. For the same line $\varphi = 45°$ and therefore $\tan \varphi = 1$. Therefore, with the unit chosen $k = 1$ and $v = \tan \varphi$.

[1] This statement by Mr. Putnam is easily verified but perhaps not entirely evident. It will be found that the time and distance of an event can be represented as coördinates with axes bisecting the angles formed by the t_1 and t_2 axes. That is, time is measured along A's line, while the distance of an event is the distance from this line of the point which represents the event, (*Relativity and Modern Physics*, page 22). Hence the speed of a particle, *i.e.*, the speed with which it is receding from A, is proportional to the slope of its line in the coördinate system formed by these axes.

Let us now consider the addition of velocities.

Assume that A, B, and C start from the same epoch with different velocities. Let

v_1 = velocity of B with reference to A,
v_2 = velocity of C with reference to B,
v_3 = velocity of C with reference to A.

Let

OA be A's line on A's chart,
OB be B's line on A's chart,
OC be C's line on A's chart.

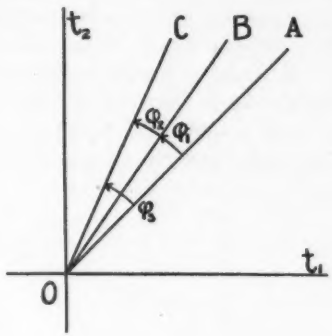

FIG. 3.

Let angle $AOB = \varphi_1$, $BOC = \varphi_2$, $AOC = \varphi_3$. Then $v_1 = \tan \varphi_1$, $v_3 = \tan \varphi_3$.

For v_2, two entirely distinct values are suggested by A's chart. First $v_2 = v_3 - v_1$, second $v_2 = \tan \varphi_2 = \tan (\varphi_3 - \varphi_1)$. These two values are about equally plausible but they are entirely different from each other. As Einstein shows v_2, the relative velocity of B and C, can be ascertained only from the chart of either B or C.

For this purpose a means of transformation from A's chart to B's is necessary.

As A is any particle we can draw a similar chart for B. But as B is moving to the right faster than A the rays of light moving to the right passing A at intervals of one second will pass B at intervals of more than a second and fewer such rays will pass B in a given time.

If t_1 is the time for A at which a given ray going to the right passes A, and t_2 the time for B at which the same ray passes B, then $t_1' = k_1 t_1$, where k_1 is a constant multiplier; and, by the same reasoning, if t_2 is the time at which a given ray going to the left passes A and t_2' is the time at which the same ray passes B, then $t_2' = k_2 t_2$ where k_2 is another constant multiplier.[1]

Let us now take any event P having on A's chart the coördinates t_1, t_2.

Draw the line OP making an angle θ with the horizontal axis.

Now show the same event on B's chart calling it P' with coördinates t_1', t_2', and draw the line OP' making an angle of θ' with the horizontal axis.

That is, $t_1' = k_1 t_1$ and $t_2' = k_2 t_2$, and $t_2'/t_1' = k_2 t_2 / k_1 t_1$. But $t_2/t_1 = \tan \theta$ and $t_2'/t_1' = \tan \theta'$. Therefore $\tan \theta' = (k_2/k_1) \tan \theta$, where k_2/k_1 is the same for all values of θ.

If β is the angle which B's line on A's chart makes with the horizontal axis, and if we take our event P on B's line, then $\theta = \beta$.

On B's chart B's line makes an angle of 45° with the horizontal axis so that $\theta' = 45°$.

In this particular case therefore the equation $\tan \theta' = (k_2/k_1) \tan \theta$ becomes $\tan 45° = (k_2/k_1) \tan \beta$, or $k_2/k_1 = \cot \beta$.

[1] This statement by Mr. Putnam is also easily verified but perhaps not entirely self-evident. (*Relativity and Modern Physics*, pages 26–29.)

Therefore any particle, whose life line on A's chart passes through the epoch and makes the angle θ with the horizontal, will on B's chart have a life line making an angle θ' with the horizontal, where $\tan \theta' = \cot \beta \tan \theta$.

Applying this formula to find the position of A's line on B's chart we have $\theta = 45°$, $\tan \theta = 1$, so that $\tan \theta' = \cot \beta$ and $\theta' = 90° - \beta$.

If we let φ be the angle between A's line and B's line on A's chart and φ' be the angle between B's line and A's line on B's chart, then $\varphi = \beta - 45°$, $\varphi' = \theta' - 45°$, from which $\varphi' = (90° - \beta) - 45° = 45° - \beta = -\varphi$. That is, the angle between A's and B's line is the same on both charts.

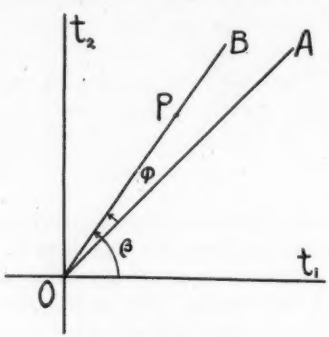

FIG. 4.

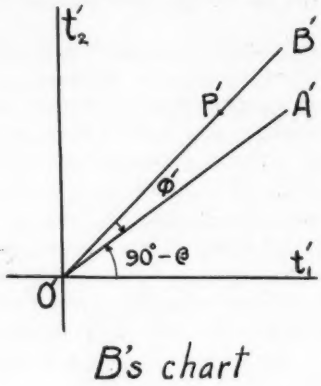

FIG. 5.

Let us now take three bodies whose lines pass through the epoch, shown on A's chart as OA, OB, OC whose angles with the horizontal are respectively 45°, β and γ.

Let the relative velocity of B to A be v_1, and of C to A, v_3. Then

$$v_1 = \tan (\beta - 45°) = \frac{\tan \beta - 1}{\tan \beta + 1},$$

$$v_3 = \tan (\gamma - 45°) = \frac{\tan \gamma - 1}{\tan \gamma + 1}.$$

From the first of these we deduce

$$\tan \beta = \frac{1 + v_1}{1 - v_1},$$

and from the second

$$\tan \gamma = \frac{1 + v_3}{1 - v_3}.$$

If v_2 is the relative velocity of C to B, to ascertain it we must transpose to B's

chart. On this chart let γ' be the angle of C's line with the horizontal then

$$v_2 = \tan(\gamma' - 45°) = \frac{\tan\gamma' - 1}{\tan\gamma' + 1}.$$

But $\tan\gamma' = \cot\beta \tan\gamma$, so that

$$v_2 = \frac{\cot\beta\tan\gamma - 1}{\cot\beta\tan\gamma + 1} = \frac{\tan\gamma - \tan\beta}{\tan\gamma + \tan\beta}$$
$$= \frac{(1+v_3)(1-v_1) - (1-v_3)(1+v_1)}{(1+v_3)(1-v_1) + (1-v_3)(1+v_1)} = \frac{v_3 - v_1}{1 - v_1v_3},$$

from which Einstein's equation follows,

$$v_3 = \frac{v_1 + v_2}{1 + v_1v_2}.$$

It is obvious that Newton's equation $v_3 = v_2 + v_1$ is inconsistent with our chart and therefore with our understanding of the Michelson-Morley discovery.

When we remember that our unit of velocity is 183,000 miles per second we see that all ordinary velocities are such small fractions that their cubes may be neglected and Einstein's equation reduces to Newton's.

CERTAIN RECURSION FORMULAS CONNECTED WITH THE SOLUTION OF $X^2 + Y^2 = Z^2$.

By P. H. DAUS, University of California, Southern Branch.

1. Introduction. In the MONTHLY for 1914, Professor G. A. Osborne considered the relations connecting the different solutions of

$$x^2 + y^2 = z^2 \qquad \text{when} \qquad y = x + 1. \tag{A}$$

Let us consider the case when

$$y = m + x, \quad m, \text{ a positive integer.} \tag{B}$$

The actual solution of the equation subject to this condition has been considered by Fermat, Frenicle, T. Pepin and others. (See Dickson's *History of the Theory of Numbers*, vol. II, pp. 183–184.) The purpose of this paper is to show how each solution can be determined in terms of the three preceding ones, indicating a marked similarity to ternary continued fractions.

Without loss of generality, we may confine our attention to the case when x, y, z are relatively prime.

2. Sets of Solutions. If in

$$x^2 + y^2 = z^2 \tag{1}$$

we put $y = x + m$ and solve for x, we get

$$x = \frac{-m \pm \sqrt{2z^2 - m^2}}{2}.$$

In order that x shall be rational, it is necessary that $2z^2 - m^2 = r^2$, which gives

$$x = (r - m)/2, \qquad y = (r + m)/2, \qquad r = x + y = 2x + m, \tag{2}$$

using the positive sign with the radical. (It will be noted that the use of the negative sign is equivalent to interchanging x and y and changing the signs of both.) That is, we must determine z and r, so that

$$2z^2 - r^2 = m^2. \tag{3}$$

From the theory of quadratic forms,[1] it follows that we must find a substitution which transforms the form $(2, 0, -1)$ into (m^2, l, n), where $l^2 \equiv 2 \pmod{m^2}$. If m is a product of k distinct primes, each prime must be of the form $8t \pm 1$, and this congruence has 2^k incongruent solutions (M, p. 36), giving 2^k sets of solutions, and this theory determines methods for finding these solutions, the actual calculations being most readily made by means of continued fractions.[2]

3. Relations between Solutions of a Set. Let us suppose that a solution (z, r) of (3) is known. Then it follows (M, pp. 88–91) that all the solutions belonging to its set will be of the form (z_n, r_n) where $(r_n + z_n\sqrt{2}) = (r + z\sqrt{2})(t + u\sqrt{2})^n$ and (t, u) is the smallest positive solution of $t^2 - 2u^2 = +1$, viz.: $t = 3$; $u = 2$.

We shall find it convenient to indicate the solutions of one set by even subscripts and those of another by odd subscripts. From

$$(r_{2n+2} + z_{2n+2}\sqrt{2}) = (r_{2n} + z_{2n}\sqrt{2})(3 + 2\sqrt{2}),$$

it follows that $r_{2n+2} = 3r_{2n} + 4z_{2n}$ and $z_{2n+2} = 2r_{2n} + 3z_{2n}$, from which we find, using (2),

$$\begin{aligned} x_{2n+2} &= 3x_{2n} + m + 2z_{2n}, \\ y_{2n+2} &= 3x_{2n} + 2m + 2z_{2n}, \\ z_{2n+2} &= 4x_{2n} + 2m + 3z_{2n}. \end{aligned} \tag{4}$$

We may note here, since from (3) r must be odd, and m is odd, that x and y are integers.

If we combine the equations of (4) for different subscripts, we will obtain (where s is replaced by x, y, z in turn)

$$s_{2n+6} = 7s_{2n+4} - 7s_{2n+2} + s_{2n}. \tag{5}$$

[1] Mathews, *Theory of Numbers*, pp. 58 and 65. Future references to this book will be indicated by (M, p. —).

[2] See note to example No. 2 at the end of this paper.

For example

$$\begin{aligned} x_{2n+4} &= 3x_{2n+2} + m + 2z_{2n+2} \\ &= 3x_{2n+2} + m + 8x_{2n} + 4m + 6z_{2n} \\ &= 3x_{2n+2} + 5m + 8x_{2n} + 3x_{2n+2} - 9x_{2n} - 3m \\ &= 6x_{2n+2} - x_{2n} + 2m. \end{aligned}$$

Likewise $x_{2n+6} = 6x_{2n+4} - x_{2n+2} + 2m$ and by subtracting

$$x_{2n+6} = 7x_{2n+4} - 7x_{2n+2} + x_{2n}. \tag{5'}$$

Similarly (5) can be established for y and z, as well as the related expressions written for odd subscripts, viz.:

$$\begin{aligned} x_{2n+3} &= 3x_{2n+1} + m + 2z_{2n+1}, \\ y_{2n+3} &= 3x_{2n+1} + 2m + 2z_{2n+1}, \\ z_{2n+3} &= 4x_{2n+1} + 2m + 3z_{2n+1}, \end{aligned} \tag{6}$$

$$s_{2n+5} = 7s_{2n+3} - 7s_{2n+1} + s_{2n-1}. \tag{7}$$

4. Arrangement of Initial Solutions. Let x_0, y_0, z_0 be the solution of

$$x^2 + y^2 = z^2, \qquad y = x + m, \tag{1}$$

obtained from the smallest value of z of a set. We will call this the fundamental solution and designate it by s_0. We obtain s_1 as follows: Determine d so that $(x_1y_1z_1)$, defined as $(x_0 + d, y_0 + d, z_0 + d)$, is a solution of (1). This is always possible in integers, for

$$(x_0 + d)^2 + (y_0 + d)^2 = (z_0 + d)^2$$

gives

$$d = 2(z_0 - x_0 - y_0) = 2(x_1 + y_1 - z_1). \tag{8}$$

Now define s_{-n} by $x_{-n} = -y_{n-1}$; $y_{-n} = -x_{n-1}$; $z_{-n} = +z_{n-1}$. We indicate the initial array below:

$$\begin{array}{cccc} -y_1 & -x_1 & z_1 & s_{-2} \\ -y_0 & -x_0 & z_0 & s_{-1} \\ x_0 & y_0 & z_0 & s_0 \\ x_1 & y_1 & z_1 & s_1. \end{array}$$

We will show that all solutions of (1) can be obtained in k sets from the three preceding ones by certain recursion formulas, which we write (s being replaced by x, y, z in turn)

$$s_{2n+1} = ps_{2n} - ps_{2n-1} + s_{2n-2}, \tag{9}$$

$$s_{2n+2} = qs_{2n+1} - qs_{2n} + s_{2n-1}, \tag{10}$$

for all values of n, where p and q, not necessarily integers, are

$$p = \frac{x_1 + y_1}{x_0 + y_0} \quad (11); \qquad q = \frac{2(x_0 + y_0 + z_0)}{d}. \quad (12)$$

5. Determination of the First Solutions. In order to establish our general results by mathematical induction, it will be necessary to determine the relations connecting s_1, s_2, s_3 with the preceding solutions.

As $y = m + x$, it is evident that, if (9) and (10) are true for the x's, they will be true for the y's. It is readily seen that, since $z_{-1} = z_0$, (9) is true for the z's, when $n = 0$, independent of p. If we are also to have (9) true for the x's,

$$x_1 = p(x_0 + y_0) - y_1 \qquad \text{or} \qquad p = \frac{x_1 + y_1}{x_0 + y_0}. \quad (11)$$

If (10) is true for $n = 0$,

$$x_2 = q(x_1 - x_0) - y_0 = qd - y_0; \qquad y_2 = qd - x_0; \qquad z_2 = qd + z_0; \quad (13)$$

must satisfy (1), *i.e.*, $(qd - y_0)^2 + (qd - x_0)^2 = (qd + z_0)^2$. Solving for q, using the fact that s_0 is a solution of (1), we get

$$q = \frac{2(x_0 + y_0 + z_0)}{d}. \quad (12)$$

We note that x_2, y_2, z_2 are integers, since qd is an integer, and also that x_2, y_2, z_2 can be written in the form of (4), identifying s_2 as obtained here with s_2 in **3**.

Now let $s_3 = p(s_2 - s_1) + s_0$. We will show that $(x_3y_3z_3)$ is a solution of (1). First we will show that $x_2 - x_1$, $y_2 - y_1$, $z_2 - z_1$ are divisible by $x_0 + y_0$, which will make x_3, y_3, z_3 integers. $x_2 - x_1 = qd - y_0 - x_1 = qd - x_0 - y_0 - d = 2(x_0 + y_0 + z_0) - x_0 - y_0 - 2(z_0 - x_0 - y_0)$ or $x_2 - x_1 = 3(x_0 + y_0) = y_2 - y_1$. Similarly $z_2 - z_1 = qd + z_0 - z_1 = qd - d = 2(x_0 + y_0 + z_0) - 2(z_0 - x_0 - y_0) = 4(x_0 + y_0)$. This gives

$$x_3 = 3(x_1 + y_1) + x_0; \qquad y_3 = 3(x_1 + y_1) + y_0; \qquad z_3 = 4(x_1 + y_1) + z_0. \quad (14)$$

Equations (14) can be written in the form of (6). For example

$$x_3 = 3x_1 + 3y_1 + x_0 = 4x_1 + 3y_1 - d = 4x_1 + 3y_1 - 2x_1 - 2y_1 + 2z_1,$$

i.e., $x_3 = 3x_1 + m + 2z_1$; and similarly for y_3 and z_3. This identifies s_3 as defined here with s_3 as obtained in **3** from s_1, and shows that s_3 is a solution of (1), since s_1 is.

6. Let us define

$$s_{2n+2}' = q(s_{2n+1} - s_{2n}) + s_{2n-1}, \quad (15)$$

$$s_{2n+3}' = p(s_{2n+2} - s_{2n+1}) + s_{2n}. \quad (16)$$

We will establish the equivalence of s as defined in **3**, with s' as defined here,

by using equations (4) and (6). This equivalence has already been established for s_2 and s_3, and it is readily established for s_1.

$$x_1 = x_0 + d = x_0 + 2(z_0 - x_0 - y_0) = -3y_0 + m + 2z_0 = 3x_{-1} + m + 2z_{-1},$$
$$y_1 = y_0 + d = y_0 + 2(z_0 - x_0 - y_0) = -3y_0 + 2m + 2z_0 = 3x_{-1} + 2m + 2z_{-1},$$
$$z_1 = z_0 + d = z_0 + 2(z_0 - x_0 - y_0) = -4y_0 + 2m + 3z_0 = 4x_{-1} + 2m + 3z_{-1}.$$

This establishes the equivalence of s_1 as defined here with that as defined by equation (6).

Now let us assume the equivalence has been established for all values up to and including s_{2n+1}. Then, using (6) and (15),

$$\begin{aligned} x_{2n+2}' &= q(x_{2n+1} - x_{2n}) + x_{2n-1} \\ &= q(3x_{2n-1} + m + 2z_{2n-1} - 3x_{2n-2} - m - 2z_{2n-2}) + x_{2n-1} \\ &= 3[q(x_{2n-1} - x_{2n-2}) + x_{2n-3}] + x_{2n+1} - 3x_{2n-1} - 2z_{2n-1} \\ &\qquad + 2[q(z_{2n-1} - z_{2n-2}) + z_{2n-3}] \\ &= 3x_{2n} + m + 2z_{2n} \text{ (using } x_{2n-1} = 3x_{2n-3} + m + 2z_{2n-3}) \\ &= x_{2n+2}; \\ z_{2n+2}' &= q(z_{2n+1} - z_{2n}) + z_{2n-1} \\ &= q(4x_{2n-1} + 2m + 3z_{2n-1} - 4x_{2n-2} - 2m - 3z_{2n-2}) + z_{2n-1} \\ &= 4[q(x_{2n-1} - x_{2n-2}) + x_{2n-3}] + z_{2n+1} - 4x_{2n-3} - 3z_{2n-3} \\ &\qquad + 3[q(z_{2n-1} - z_{2n-2}) + z_{2n-3}] \\ &= 4x_{2n} + 2m + 3z_{2n} = z_{2n+2}. \quad \text{Likewise } y_{2n+2}' = y_{2n+2}. \end{aligned}$$

By advancing the subscripts one and changing q to p, we establish the corresponding relations for odd subscripts. It follows then that the solutions as defined by (15) and (16) are equivalent to the solutions defined by (4) and (6), or their equivalents (5) and (7).

7. Number of Expansions. We see then that each expansion, defined by equations (9) and (10) in **4**, accounts for two sets of congruent solutions of **2**. It follows then that if m is a prime or a power of a prime of the proper form, our expansion in **4** gives all the solutions, while if m has k distinct prime factors there will be k expansions. It is to be noted likewise, since the results of section **4** did not depend upon which solution we took as s_0, *that we can begin with any solution and will always get the same group to which it belongs*, the difference merely being the relative positions of the two subgroups of which it is composed. In particular, when m is a prime or a power of a prime, we get all solutions if we start with any one. This is illustrated by example 3 below. We conclude with

8. Numerical Examples. EXAMPLE 1.

If $m = 1$, $l^2 \equiv 2 \pmod 1$ really has only one solution and this leads to the trivial solution (0, 1, 1). No advantage is gained by the use of (9) and (10), while all solutions are readily obtained by finding s_2 and s_3 from (4) and then using (5).

	x	y	z	
	0	1	1	s_1
	3	4	5	s_2
	20	21	29	s_3
− 7, 7	119	120	169	s_4
− 7, 7	696	697	985	s_5
− 7, 7	4059	4060	5741	s_6
		etc.		

EXAMPLE 2. $m = 7$.

The congruence $l^2 \equiv 2 \pmod{49}$ has the solutions $l \equiv \pm 10 \pmod{49}$. It will be noticed, if $(2, 0, -1)\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} (m^2, l, n)$, that $(2, 0, -1)\begin{pmatrix} \alpha & -\beta \\ -\gamma & \delta \end{pmatrix} (m^2, -l, n)$. The difference between the two solutions is that the first yields $z = \alpha$, $r = \gamma$; the second $z = \alpha$, $r = -\gamma$, which is equivalent to using the negative sign with the radical. It will be convenient to use $l \equiv -10$, because the principal root of $(49, -10, 2)$ is positive. It is then found [1]

$$(2, 0, -1)\begin{pmatrix} 5 & -1 \\ 1 & 0 \end{pmatrix} (49, -10, 2), \qquad \text{or} \qquad z = 5, \quad r = 1.$$

This leads to $s_0 = (-3, 4, 5)$ and $d = 8$; $s_1 = (5, 12, 13)$, $p = 17$, $q = 3/2$, and we indicate the first 8 solutions below:

	x	y	z	
	− 12	− 5	13	s_{-2}
	− 4	3	5	s_{-1}
	− 3	4	5	s_0
− 17, 17	5	12	13	s_1
− 3/2, 3/2	8	15	17	s_2
− 17, 17	48	55	73	s_3
− 3/2, 3/2	65	72	97	s_4
− 17, 17	297	304	425	s_5
− 3/2, 3/2	396	403	565	s_6
− 17, 17	1748	1755	2477	s_7
− 3/2, 3/2	2325	2332	3293	s_8.
		etc.		

The division into two groups is indicated below:

GROUP I.					GROUP II.				
	− 15	− 8	17	s_{-3}		− 12	− 5	13	s_{-2}
	− 4	3	5	s_{-1}		− 3	4	5	s_0
	5	12	13	s_1		8	15	17	s_2
− 7, 7	48	55	73	s_3	− 7, 7	65	72	97	s_4
− 7, 7	297	304	425	s_5	− 7, 7	396	403	565	s_6
− 7, 7	1748	1755	2477	s_7	− 7, 7	2325	2332	3293	s_8
		etc.					etc.		

EXAMPLE 3. If we start with any other solution, as s_1' (say s_4), *we will get all the solutions*, but the two sets in example 2 will be in different relative positions. $s_1' = s_4 = (65, 72, 97)$ gives $s_0' = (-15, -8, 17)$, after we have first found $d = 80$. $p' = -137/23$; $q' = -3/20$. The expansion can readily be formed by the reader. He will find the solutions appear in the following order, s_{-5}, s_2, s_{-3}, s_4, s_{-1}, s_6, s_1, s_8, etc.

[1] This may be found conveniently by the use of the following lemma: If the principal root of (a, b, c) be expanded into a continued fraction, and if $\frac{\sqrt{D} + P_n}{Q_n}$ and $\frac{A_n}{B_n}$ represent the nth complete quotient and convergent sets, then

$$(a, b, c)\begin{pmatrix} A_n(-1)^n A_{n+1} \\ B_n(-1)^n B_{n+1} \end{pmatrix} ((-1)^n Q_{n+1}, P_{n+2}, (-1)^{n+1} Q_{n+2}).$$

Application of this lemma to the roots of $(2, 0, -1)$ and $(49, -10, 2)$ gives the desired substitution, which might have been written by inspection in this simple example.

ON THE THEORY OF EQUATIONS FROM THE STANDPOINT OF VECTOR ANALYSIS.

By PETER FIELD, University of Michigan.

The problem of solving an algebraic equation requires that a number which satisfies a certain condition shall be determined. If a graphical solution is attempted, the case of real roots requires a different construction from that for imaginary roots. The advantage of viewing the problem from the standpoint of vector analysis lies in the fact that it gives a geometrical point of view which holds equally well for real or imaginary roots and for real or imaginary coefficients. The idea involves simply the elementary properties of complex numbers.

Let $\mathbf{t}$ be a given unit vector, let a_j be a known operator which when operating on any vector multiplies the modulus of the vector by a positive number r_j and turns the vector through an angle φ_j in a given plane, and let x be an unknown operator of the same nature as a_j, the corresponding multiplier and angle being r and φ. In the notation of Burali-Forti and Marcolongo, *Éléments de Calcul Vectoriel*, $a_j = r_j e^{i\varphi_j}$, $x = re^{i\varphi}$. The equation

$$(a_n x^n + a_{n-1}x^{n-1} + \cdots a_0)\mathbf{t} = 0, \tag{1}$$

expresses the condition that x shall be so chosen that the polygon formed by the $n+1$ vectors $a_j x^j \mathbf{t}$ shall close. If φ_j is restricted to the values 0 and π, the polygon will have the angle between its consecutive sides φ or $\pi - \varphi$. This corresponds to the equation with real coefficients. In this case if $x = re^{i\varphi}$ is a solution of (1), so is $x = re^{-i\varphi}$. The second polygon differs from the first only in the sense of rotation. In other words, in an equation with real coefficients the imaginary roots enter in pairs. Corresponding to the case of an equation with real coefficients and a real root, there results a polygon whose vertices all lie on the same line.

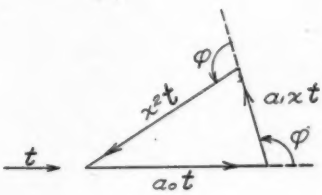

FIG. 1.

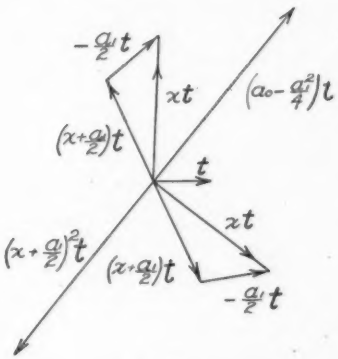

FIG. 2.

Consider in particular the case corresponding to a quadratic equation with

positive real coefficients and imaginary roots. The vector triangle $(x^2+a_1x+a_0)t$ is isosceles (Fig. 1). Therefore $x = re^{i\varphi}$ where $r = \sqrt{a_0}$, $\varphi = \arccos(-a_1/2\sqrt{a_0})$ and the vector xt is readily constructed. The most simple construction which is applicable in all cases appears to result from considering the equation in the form

$$\left[\left(x+\frac{a_1}{2}\right)^2 + a_0 - \frac{a_1^2}{4}\right]t = 0.$$

The two values of xt are indicated in Fig. 2.

A SUGGESTED LIST OF MATHEMATICAL BOOKS FOR JUNIOR COLLEGE LIBRARIES.[1]

I. ELEMENTARY BOOKS.[2]

Algebra.

DICKSON, L. E. First Course in the Theory of Equations. 1922. $1.75. Wiley.

FINE, H. B. College Algebra. $2.40. Ginn.

HALL, H. S. and KNIGHT, S. R. Higher Algebra (Revised and enlarged by F. L. Sevenoak). $1.60. Macmillan.

LIPKA, J. Graphical and Mechanical Computation. 1921. $4.00. Wiley.

SMITH, C. A Treatise on Algebra. $3.00. Macmillan.

Geometry.

ALTSHILLER-COURT, N. College Geometry. $4.00. Johnson Pub. Co.

CARSLAW, H. S. Elements of Non-Euclidean Plane Geometry and Trigonometry. $1.80. Longmans, Green & Co.

GODFREY, C. and SIDDONS, A. W. Modern Geometry. $2.00. Macmillan.

HALSTED, G. B. Rational Geometry (2d edition). $1.75. Wiley.

KENISON, E. and BRADLEY, H. C. Descriptive Geometry. $2.25. Macmillan.

LEHMER, D. N. Synthetic Projective Geometry. $1.28. Ginn.

SMITH, C. and BRYANT, S. Euclid's Elements of Geometry. London, 1904. Bks. I–IV. London. 1904. 3 s. Macmillan.

SMITH, C. Geometrical Conics. 1894. 7 s. Macmillan.

Analytic Geometry.

OSGOOD, W. F. and W. C. GRAUSTEIN. Plane and Solid Analytic Geometry. 1921. $3.00. Macmillan.

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[1] This list was prepared by Professors B. A. BERNSTEIN, F. CAJORI, E. R. HEDRICK, C. A. NOBLE and T. M. PUTNAM, of the University of California.

A similar list was prepared by a committee of the Association and was printed in the MONTHLY (*1917*, 368–376). EDITOR.

[2] The classification "elementary" and "advanced" is made on the supposition that an "advanced" book is one that is designed for readers having considerable mathematical equipment or considerable maturity of mind.

SMITH, C. An Elementary Treatise on Conic Sections. Revised and enlarged ed., 1910. $2.75. Macmillan.

SMITH, C. An Elementary Treatise on Solid Geometry. 1907. $3.50. Macmillan.

Calculus.

GIBSON, G. A. Elementary Treatise on the Calculus. 1906. $2.90. Macmillan.

OSGOOD, W. F. Introduction to the Calculus. 1922. $2.90. Macmillan.

PEIRCE, B. O. A Short Table of Integrals. (2d revised edition $1.48, Abridged $.40.) Ginn.

PHILLIPS, H. B. Differential Equations. $1.50. Wiley.

THOMPSON, S. P. Calculus made easy. 2d edition. 1916. $1.00. Macmillan.

History.

BALL, W. W. R. A short History of Mathematics. $4.00. Macmillan.

CAJORI, F. History of Elementary Mathematics. 2d edition, 1917. $2.25. Macmillan.

CAJORI, F. History of Mathematics. 2d edition, 1919. $4.00. Macmillan.

HEATH, T. L. History of Greek Mathematics. 2 Vols. $16.70. Clarendon Press, Oxford.

KARPINSKI, L. C. History of Arithmetic. 1925. $2.00. Rand McNally.

SMITH, D. E. History of Mathematics. Vol. I, 1923. $4.00. Vol. II, 1925. $4.40. Ginn.

SMITH, D. E. and KARPINSKI, L. C. The Hindu-Arabic Numerals. 1911. $1.60. Ginn.

TROPFKE, J. Geschichte der Elementar-Mathematik. Parts I–VII, 1921–1924. Berlin. $8.50. W. deGruyter & Co.

Miscellaneous.

(ABBOTT, E. A.). Flatland. By A. Square. $1.25. Little, Brown & Co.

BALL, W. W. R. Mathematical Recreations. 7th edition. $3.50. Macmillan.

BALL, W. W. R. Introduction to String Figures. 1924. 2 s. 6 p. Heffer & Sons.

BOON, F. C. Companion to Elementary School Mathematics. 1924. $4.50. Longmans, Green & Co.

BRUHNS, C. C. A new Manual of Logarithms to Seven Places of Decimals. 1870. $2.25. Lemcke & Buechner, N. Y. (Stechert, N. Y., dealer.)

CARSON, G. ST. L. Essays on Mathematical Education. $1.20. Ginn & Co.

DAVISON, C. Subjects for Mathematical Essays. $1.60. Macmillan.

DEMORGAN, A. A Budget of Paradoxes. 2 Vols. $5.00. Open Court Publishing Co.

DUDENEY, H. E. The Canterbury Puzzles. 1908. $2.00. E. P. Dutton.

FORSYTH, C. H. Mathematical Theory of Life Insurance. $1.25. Wiley.

FORSYTH, C. H. Mathematical Analysis of Statistics. $2.25. Wiley.
HART, W. L. Mathematics of Investment. 1924. $2.50. With tables $3.40. Heath.
HOBSON, E. W. Treatise on Plane Trigonometry. 4th Edition. $5.00. Macmillan.
KARSTEN, K. G. Charts and Graphs. 1923. $6.00. Prentice-Hall.
KELLEY, T. J. Statistical Method. 1923. $4.00. Macmillan.
LENNES, N. J. The Teaching of Arithmetic. 1923. $2.00. Macmillan.
MACFARLANE, A. Ten British Mathematicians of the Nineteenth Century. $1.50. Wiley.
MANNING, H. P. Fourth Dimension Simply Explained. 1922. $1.50. Scientific American Pub. Co.
MOULTON, F. R. Introduction to Astronomy. $3.25. Macmillan.
PUTNAM, T. M. Mathematical Theory of Finance. 1924. $1.75. Wiley.
RIETZ, H. L., CRATHORNE, A. R. and RIETZ, J. C. Mathematics of Finance. 1921. $3.00. Holt.
ROW, T. S. Geometric Paper Folding. Edition by W. W. Beman and D. E. Smith. $1.00. Open Court Pub. Co.
SCHUBERT, H. Mathematical Recreations. $.75. Open Court Pub. Co.
SCHULTZE, A. The Teaching of Mathematics in Secondary Schools. $1.80. Macmillan.
SKINNER, E. B. Mathematical Theory of Investment. Revised edition, 1925. $3.40. Ginn.
SMITH, D. E. The Teaching of Geometry. $1.40. Ginn.
SMITH, D. E. Number Stories of Long Ago. 1919. $.60. Ginn.
VON SANDEN, H. Practical Mathematical Analysis. Translation by H. Levy. 1923. 12 s. 6 d. Methuen.
WELD, LEROY D. Theory of Errors and Least Squares. 1916. $2.00. Macmillan.
WELD, L. G. Determinants. 4th edition, monograph series no. 3, 1906. $1.25. Wiley.
WHITTAKER, E. T., and ROBINSON, G. A Short Course in Interpolation. 5 s. Blackie & Sons.

II. ADVANCED BOOKS.

Algebra.

BÔCHER, M. Introduction to Higher Algebra. 1907. $3.00. Macmillan.
BURNSIDE, W. S., and PANTON, A. W. Theory of Equations. Vol. I, $5.00; Vol. II, $5.00. Longmans, Green & Co.
FINE, H. B. The Number System of Algebra. $1.40. Heath.
GLENN, O. E. Theory of Invariants. 1915. $3.25. Ginn.
MURNAGHAN, F. D. Vector Analysis and Relativity. 1922. $2.75. Johns Hopkins Press.
RUNNING, T. R. Empirical Formulas. 1917. $2.00. Wiley.

WEBER, H. Lehrbuch der Algebra (small edition). 1912. About 15 gold marks. Vieweg und Sohn.

Geometry.

BELL, R. J. T. Coordinate Geometry of Three Dimensions. 2d Edition, 1923. $4.25. Macmillan.

COOLIDGE, J. L. Geometry of the Complex Domain. $6.00. Oxford Univ. Press.

COOLIDGE, J. L. The Elements of Non-euclidean Geometry. $5.35. Oxford Univ. Press.

CREMONA, L. Elements of Projective Geometry. $5.00. Clarendon Press.

DOWLING, L. W. Projective Geometry. 1917. $2.00. McGraw-Hill.

DURELL, C. V. Modern Geometry. 1920. $2.00. Macmillan.

FRANKLAND, W. B. Theories of Parallelism. $2.25. Cambridge University Press (Macmillan).

HEATH, T. L. The Thirteen Books of Euclid's Elements. 3 Vols. Set: $21.00. Macmillan.

HILBERT, D. The Foundations of Geometry. Translation by Townsend. $1.00 cloth. Open Court Pub. Co.

HILTON, H. Plane Algebraic Curves. 1920. $9.35. Clarendon Press.

LOBACHEVSKI, N. I. Theory of Parallels. Translation by Halsted. $1.25. Open Court Pub. Co.

MANNING, H. P. Geometry of Four Dimensions. 1914. $3.00. Macmillan.

MANNING, H. P. Non-Euclidean Geometry. $1.36. Ginn.

MATHEWS, G. B. Projective Geometry. $1.80. Longmans, Green & Co.

NEVILLE, E. H. Prolegomena to Analytical Geometry. 1922. $10.50. Cambridge Univ. Press.

NEVILLE, E. H. The Fourth Dimension. $1.60. Cambridge Univ. Press.

SOMERVILLE, D. M. Y. Elements of Non-Euclidean Geometry. 7 s. 6 d. Harcourt Brace & Co.

SNYDER, V. and SISAM, C. R. Analytic Geometry of Space. $3.00. Holt.

WINGER, R. M. Introduction to Projective Geometry. $4.40. Heath.

WOODS, F. S. Higher Geometry. $5.00. Ginn.

Calculus and Differential Equations.

BATEMAN, H., Differential Equations. 1918. $3.25. Longmans, Green & Co.

COHEN, A. Differential Equations. $2.40. Heath.

FORSYTH, A. R. Treatise on Differential Equations. 4th Edition. $6.00. Macmillan.

HEDRICK, E. R. and KELLOGG, O. D. Applications of the Calculus to Mechanics. $1.36. Ginn.

JOHNSON, W. W. A Treatise on Ordinary and Partial Differential Equations. $3.50. Wiley.

MURRAY, D. A. Differential Equations. $2.25. Longmans, Green & Co.

WILSON, E. B. Advanced Calculus. 1912. $5.00. Ginn.

Miscellaneous.

BIRKHOFF, G. D. Lowell Lectures on Relativity. Macmillan.

BIRKHOFF, G. D. Relativity and Modern Physics. 1923. $4.00. Harvard Univ. Press.

BLISS, G. A. Calculus of Variations. 1925. $2.00. Open Court Pub. Co.

BURKHARDT, H. Theory of Analytic Functions. Translation by Rasor. $4.40. Heath.

BYERLY, W. E. Fourier Series. $3.00. Ginn.

BYERLY, W. E. Harmonic Functions. 4th Edition, Monograph Series No. 5. $1.25. Wiley.

CAJORI, F. Theory of Equations. 1904. $2.00. Macmillan.

CANTOR, G. Contributions to the Founding of the Theory of Transfinite Numbers. Translation by Jourdain. $1.25. Open Court Pub. Co.

CARMICHAEL, R. D. Diophantine Analysis. Monograph No. 16. $1.50. Wiley.

CARMICHAEL, R. D. Theory of Numbers. $1.25. Wiley.

Chambers' Mathematical Tables. Edited by J. Pryde. $2.50. Van Nostrand.

COFFIN, J. G. Vector Analysis. 2d Edition. $2.50. Wiley.

COHEN, A. The Lie Theory of One-parameter Groups. $2.40. Heath.

COOLIDGE, J. L. Treatise on the Circle and the Sphere. 1916. $8.35. Clarendon Press.

COUTURAT, L. Algebra of Logic. Translation by Robinson. $1.50. Open Court Pub. Co.

CURTISS, D. R. Theory of Functions. $2.00. Open Court Pub. Co.

DICKSON, L. E. Algebras and their Arithmetics. 1923. $2.25. Univ. of Chicago, Science Series.

DICKSON, L. E. Algebraic Invariants. Monograph Series No. 14. $1.50. Wiley.

DIXON, A. C. Elementary Properties of Elliptic Functions. London. 1894. 5 shillings. Macmillan.

EISENHART, L. P. A Treatise on the Differential Geometry of Curves and Surfaces. 1909. $5.00. Ginn.

FISKE, T. S. Functions of a Complex Variable. Monograph Series No. 11. $1.25. Wiley.

GOURSAT, E. Course in Mathematical Analysis. Translation by E. R. Hedrick. Vol. I, $5.00. Vol. II (Hedrick and Dunkel), Part I, $3.50; Part II, $3.50. Ginn.

HARDY, G. H. A Course in Pure Mathematics. 3d Edition. 1921. 15 s. Cambridge Univ. Press.

HOBSON, E. W. The Theory of Functions of a Real Variable. 2d Edition. Vol. I. $15.00. Macmillan.

HUNTINGTON, E. V. The Continuum and Other Types of Serial Order. $1.25. Harvard Univ. Press.

JEANS, J. H. Theoretical Mechanics. 1907. $3.50. Ginn.

KEYSER, C. J. Mathematical Philosophy. 1922. $4.70. Dutton.

KEYSER, C. J. Human Worth of Rigorous Thinking. $3.00. Columbia Univ. Press.

LONEY, S. L. An Elementary Treatise on the Dynamics of a Particle and of Rigid Bodies. 2d Edition. $4.75. Macmillan.

MACMAHON, J., Hyperbolic Functions. Monograph Series, No. 4. $1.25. Wiley.

MERRIMAN, M. and WOODWARD, R. S. Higher Mathematics. 2d Edition revised. $5.00. Wiley.

MILLER, G. A., BLICHFELDT, H. F. and DICKSON, L. E. Theory and Applications of Finite Groups. $4.00. Wiley.

NUNN, T. P. Relativity. 1923. $2.40. Dutton.

NUNN, T. P. The Teaching of Algebra, including Trigonometry. $2.75. Longmans, Green & Co.

PEARSON, K. Grammar of Science. Third edition. Part I: Physical, Part II: Biological. Macmillan.

PICARD, C. E. Traité d'Analyse. 3 Vols., 1901–1906. 2d editions: Vol. I, 1901, 16 francs; Vol. II, 1905, 18 francs; Vol. III, 1908, 18 francs. Gauthier-Villars.

PIERPONT, J. Theory of Functions of Real Variables. Two volumes. $5.00 each. Ginn.

POINCARÉ, H. The Foundations of Science. 1912–1921. Translated by Halsted. $5.00. The Science Press, New York.

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QUESTIONS AND DISCUSSIONS.

EDITED BY C. F. GUMMER, Queen's University, Kingston, Ont., Canada.

The department of Questions and Discussions in the MONTHLY is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the separate department of Problems and Solutions.

QUESTIONS.

54. The search for three-dimensional analogues of Rolle's Theorem suggests the following question: C is a simple closed curve in three dimensions, having at every point a tangent which turns continuously with respect to the arc. O is a fixed point on C, at which curvature and torsion exist and do not vanish. At no point on C is the tangent parallel to the tangent at O. Under these conditions, do there always exist points P and Q on C, such that OP is parallel to the tangent at Q? Possibly some modification of the assumptions would lead to an interesting decision.[1]

REPLIES TO QUESTIONS.

52. Is there any simple treatment of the regular pentagon constructions the proof of which involves neither medial section nor trigonometric functions?

REPLY BY H. C. BRADLEY, Massachusetts Institute of Technology.[2]

A proof, fulfilling the requirements except perhaps as to simplicity, can be obtained by representing the radius of the circle by the quantity $\sqrt{5}+1$. Only the Pythagorean theorem is necessary in the demonstration.

See Fig. 1. Let O be the center of a circle, OA any radius. Let B be the point where $OB : BA = 1 : \sqrt{5}$. Draw BC perpendicular to OA, meeting the circumference at C. Then AC is 1/5 of the circumference.

Proof. Produce the radius, and obtain the symmetrical points A', B', and C'. Then $CC' = BB'$. If $OB = 1$, then $CC' = 2$.

Bisect arc AC at D. Now, if we can show that $AD = DC = 2$, that is, $AD = CC'$, the semicircle is divided into fifths, and AC is one-fifth of the circumference.

Draw OC, OD, and AC, the latter two intersecting at E.

In $\triangle OBC$, $OB = 1$, $OC = \sqrt{5}+1$. Hence $CB = \sqrt{5+2\sqrt{5}}$.

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[1] If C contains two points at which the tangents are parallel but opposite, P and Q may be so determined in at least two ways. EDITOR.

[2] Other replies have already appeared by Norman Anning (*1925*, 75) and C. H. Chepmell (*1925*, 76). See also the discussion by H. C. Bradley (*1924*, 342). EDITOR.

In $\triangle ABC$, $AB = \sqrt{5}$, $CB = \sqrt{5 + 2\sqrt{5}}$. Hence $AC = \sqrt{10 + 2\sqrt{5}}$, $AE = \frac{1}{2}AC = \frac{1}{2}\sqrt{10 + 2\sqrt{5}}$.

In $\triangle OAE$, $OA = \sqrt{5} + 1$, $AE = \frac{1}{2}\sqrt{10 + 2\sqrt{5}}$. Hence $OE = \frac{1}{2}\sqrt{14 + 6\sqrt{5}}$, $= \frac{1}{2}(\sqrt{5} + 3)$.

$DE = OD - OE = \sqrt{5} + 1 - \frac{1}{2}(\sqrt{5} + 3) = \frac{1}{2}(\sqrt{5} - 1)$.

In $\triangle AED$, $AE = \frac{1}{2}\sqrt{10 + 2\sqrt{5}}$, $DE = \frac{1}{2}(\sqrt{5} - 1)$. Hence $AD = 2$, and the proposition is proved.

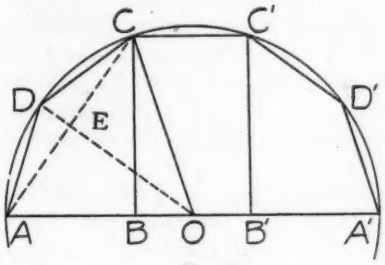

FIG. 1.

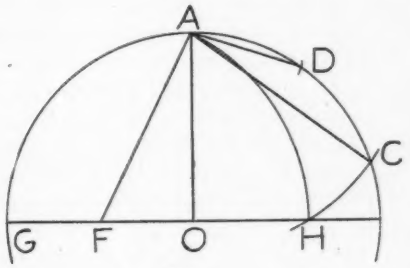

FIG. 2.

We can now prove a construction given in text-books for the sides of the decagon and pentagon. See Fig. 2. The radius OG is perpendicular to OA, and F is its mid-point.

$$OA = \sqrt{5} + 1. \qquad OF = \tfrac{1}{2}(\sqrt{5} + 1).$$

Hence

$$AF = \tfrac{1}{2}\sqrt{30 + 10\sqrt{5}} = \tfrac{1}{2}(\sqrt{5} + 5). \qquad FH = AF = \tfrac{1}{2}(\sqrt{5} + 5).$$

$OH = FH - OF = 2$ = side of decagon. Then in $\triangle AOH$, $AH = \sqrt{10 + 2\sqrt{5}}$ = side of pentagon.

DISCUSSIONS.

I. ON A METHOD OF APPROXIMATING THE REAL ROOTS OF A POLYNOMIAL.

BY G. W. SPENCELEY, Miami University.

The word "equation" throughout this paper will refer only to the integral algebraic equation

$$p_0x^n + p_1x^{n-1} + p_2x^{n-2} + \cdots p_n = 0 \qquad (1)$$

in which the coefficients are real and rational, and one root at least is real.

In such an equation, if R be a real positive root which is large in comparison

with s, the sum of all the other roots, we shall have

$$R + s = -\frac{p_1}{p_0},$$

and approximately,

$$s = -\frac{p_2}{p_1}.$$

If the approximation to s be accurate to the nth digit in the decimal, then R can be calculated from the above, correct to the same degree.

It is only one step further to the approximating of any real root of a given equation by transforming that equation into another in such a way as to make the desired root go over into a positive real root, large in comparison with the sum of all the other roots of the transformed equation. The purpose of this paper is to describe such a transformation and the method of applying it to the evaluation of any real root of a given equation.

First, we shall find the general transformation which is equivalent geometrically to reflecting all the roots, real and imaginary, in a circle with center a and radius r,—the origin being moved in the process first to the point a and then arbitrarily back a distance b, that is, equivalent to moving the origin to the point $a-b$. We may do this in three transformatious, which combine into

$$x = \frac{r^2}{y - b} + a. \tag{2}$$

In what follows a, b, and r are real.

First Step. Let x_1 be the real root of equation (1) which it is desired to approximate; and let the other $n - 1$ roots be represented by $x_2, x_3, x_4 \cdots x_n$. Let us further assume that a first approximation a_1 for the root x_1 is known, and that $a_1 < x_1$.

Now if equation (1) be subjected to the transformation (2), the center of the circle of reflection being a_1, and the radius r_1 being chosen sufficiently small to exclude from the circle all the roots, real and imaginary, except x_1, we shall obtain a new equation in y

$$q_0 y^n + q_1 y^{n-1} + q_2 y^{n-2} + \cdots + q_n = 0, \tag{3}$$

whose roots $y_1, y_2, y_3 \cdots y_n$ are the transformed values of $x_1, x_2, x_3 \cdots x_n$ respectively, and such that all the roots of (3) except y_1 will be included in the circle of reflection, y_1 being outside and to the right.

If the roots of the original equation (1) be so separated as to enable us to take $r_1 = 1$, and if at this step we take $b_1 = 0$, we obtain Lagrange's familiar transformation

$$x = (1/y) + a_1. \tag{4}$$

The transformation (2) serves the purpose of separating y_1 from the other

roots of (3), and confining the latter roots to the small region within the circle of reflection.

A first approximation to y_1 is $-q_1/q_0$. Guided by this value we can find by substitution in equation (3) an approximate value a_2 which will be less than y_1 but differ from it by less than unity.

Second Step. To increase further the value of the root y_1, and at the same time diminish the value of the sum of the remaining roots, we shall reflect the y-roots in a circle with center a_2, taking $r_2 = \sqrt{a_2}$, $b_2 = -1$, which gives us for our equation of transformation

$$y = a_2 z/(z - 1). \tag{5}$$

If the resulting equation in z be represented by

$$t_0 z^n + t_1 z^{n-1} + t_2 z^{n-2} + \cdots + t_n = 0,$$

then the sum s of the small roots $z_2, z_3, z_4 \cdots z_n$ is approximately $-t_2/t_1$, and the large root z_1 is obtained from $z_1 + s = -t_1/t_0$.

We have thus far:

$$x_1 = a_1 + \frac{1}{a_2} - \frac{1}{a_2 z_1}.$$

By continuing the process we may obtain the value of x_1 to any degree of accuracy desired.

In practice it sometimes happens that a_2 may be taken more conveniently to the right of y_1, in which case we may reflect in the circle as before, the z_1 now going far to the left and becoming negative. By setting $z = -z_1$ we can change z_1 to a positive root. The resulting transformation is given by

$$y = a_2 z/(z + 1), \tag{6}$$

a very useful addition to equation (5). In this case we should have

$$x_1 = a_1 + \frac{1}{a_2} + \frac{1}{a_2 z_1}.$$

It remains to calculate the large root in the final equation and estimate the error.

Assume the final equation to be

$$c_0 w^n + c_1 w^{n-1} + c_2 w^{n-2} + \cdots + c_n = 0, \tag{7}$$

whose large root w_1 we shall now call R, and the sum of whose small roots w_2, $w_3, w_4 \cdots w_n$ we shall write as s.

By substituting R for w in (7) and transposing terms, we get

$$c_0 R^n = -c_1 R^{n-1} - c_2 R^{n-2} - c_3 R^{n-3} - \cdots - c_n,$$

whence

$$R = -\frac{c_1}{c_0} - \frac{c_2}{c_0 R} - \frac{c_3}{c_0 R^2} - \cdots - \frac{c_n}{c_0 R^{n-1}}. \tag{8}$$

Again,

$$R + s = -c_1/c_0. \tag{9}$$

From (8) and (9) we get

$$s = \frac{c_2}{c_0 R} + \frac{c_3}{c_0 R^2} + \frac{c_4}{c_0 R^3} + \cdots + \frac{c_n}{c_0 R^{n-1}}.$$

In this equation substitute

$$R = -(c_1 + c_0 s)/c_0,$$

obtained from (9), to get finally

$$s = -\frac{c_2}{c_1\left(1 + \frac{c_0 s}{c_1}\right)} + \frac{c_0 c_3}{c_1^2\left(1 + \frac{c_0 s}{c_1}\right)^2} - \frac{c_0^2 c_4}{c_1^3\left(1 + \frac{c_0 s}{c_1}\right)^3} + \cdots.$$

On expanding this becomes

$$\begin{aligned} s = &-\frac{c_2}{c_1} + \frac{c_0 c_2 s}{c_1^2} - \frac{c_0^2 c_2 s^2}{c_1^3} + \frac{c_0^3 c_2 s^3}{c_1^4} - \frac{c_0^4 c_2 s^4}{c_1^5} + \cdots \\ &+ \frac{c_0 c_3}{c_1^2} - \frac{2 c_0^2 c_3 s}{c_1^3} + \frac{3 c_0^3 c_3 s^2}{c_1^4} - \frac{4 c_0^4 c_3 s^3}{c_1^5} + \cdots \\ &- \frac{c_0^2 c_4}{c_1^3} + \frac{3 c_0^3 c_4 s}{c_1^4} - \frac{6 c_0^4 c_4 s^2}{c_1^5} + \cdots \\ &+ \frac{c_0^3 c_5}{c_1^4} - \frac{4 c_0^4 c_5 s}{c_1^5} + \cdots \\ &- \frac{c_0^4 c_4}{c_1^5} + \cdots \\ &+ \cdots. \end{aligned} \tag{10}$$

Terms in columns are of about the same order of magnitude. It is to be observed that s, which is to be computed, appears on both sides of the equation; and that the series on the right are infinite. Excepting the numerical coefficients in the second and following series, any term after the first term in each series may be obtained from the preceding term of the same series by multiplying it by $-c_0 s/c_1$. This factor is of course equal to $s/(R+s)$, and hence a small fraction, so that these series in practice are rapidly convergent and s may be calculated by means of the first few terms, those neglected giving limits to the error in s. But to show the convergence of these series, without begging the question, it is

necessary to obtain, in an independent manner,[1] an upper limit to $|s|$ and hence to $|c_0 s/c_1|$. In practice this may be omitted and the approximate value $-c_2/c_1$ for s may be used in estimating the error due to the neglected terms. With regard to the numerical coefficients it is to be observed that they are the numbers of Pascal's triangle

1	1	1	1	1	1	1	1	1	·	·	·	$u_n = 1$
1	2	3	4	5	6	7	8	9	·	·	·	$u_n = n$
1	3	6	10	15	21	28	·	·	·	·	·	$u_n = n(n+1)/2!$
1	4	10	20	35	·	·	·	·	·	·	·	$u_n = n(n+1)(n+2)/3!$
·	·	·	·	·	·	·	·	·	·	·	·	·

These numbers increase from left to right in any row, but the ratio u_n/u_{n-1} decreases monotonically to the limit 1 as n increases and becomes infinite. Hence the convergence of the series when $|c_0 s/c_1| < 1$. If the series are alternating, a limit to the error in each is given by the first neglected term in each case. If the signs are alike, a limit to the error in each series is gotten by summing the neglected terms in each series as a geometric series, the numerical coefficient (say N) in the first neglected term of each series being included with $-c_0 s/c_1$ to form the geometric factor $-Nc_0 s/c_1$.

An upper limit to the error in s being known we can calculate R from equation (9) and set an upper limit to its error. The error in x_1 is then determined from the equation

$$x_1 = a_1 + \frac{1}{a_2} - \frac{1}{a_2 R}$$

by means of

$$\Delta x_1 = \frac{dx_1}{dR} \cdot \Delta R = \frac{\Delta R}{a_2 R^2},$$

where Δx_1 represents the error in x_1 and ΔR the error in R.

[1] An independent upper limit to $|s|$ may be obtained thus. Let the two points $(a_1 + r_1)$ and $(a_1 - r_1)$ on the x-axis be transformed finally into the points w' and w'' respectively. Let the equations of transformation be

$$x = \frac{r_1^2}{y} + a_1; \qquad y = \frac{a_2 z}{z \pm 1}; \qquad z = \frac{a_3 u}{u \pm 1}; \qquad u = \frac{a_4 v}{v \pm 1}; \qquad v = \frac{a_5 w}{w \pm 1}.$$

Then none of the small roots $w_2, w_3, w_4 \cdots w_n$ lie outside of the region bounded by w' and w''. Also the origin lies within this same region, and hence, if d represent the distance between w' and w'' taken positively, we shall have

$$d = |w' - w''|.$$

Now there are $n - 1$ such small roots in this region; hence

$$|s| \leq (n-1)d.$$

This distance d will vary according as the signs in the denominators of the equations of transformation are positive or negative; but the maximum d will be gotten by taking them all positive. In this case

$$d = \frac{2r_1}{a_2 a_3 a_4 a_5 \cdots}\left[\frac{1}{1 - r_1^2\left(\frac{1}{a_2} + \frac{1}{a_2 a_3} + \frac{1}{a_2 a_3 a_4} + \frac{1}{a_2 a_3 a_4 a_5} + \cdots\right)^2}\right].$$

A convenient approximation to s may be obtained from the first three terms of (10); whence

$$s = \frac{c_0c_3 - c_1c_2}{c_1^2 - c_0c_2}. \tag{11}$$

Example. Take

$$x^3 + 4x^2 - 7 = 0 \qquad (\text{root } x_1 = 1 +).$$

Step 1. Set

$$x = \frac{1}{y} + 1$$

to get

$$2y^3 - 11y^2 - 7y - 1 = 0 \qquad (\text{root } y_1 = 6 +).$$

Step 2. Set

$$y = \frac{6z}{z - 1}$$

to get

$$7z^3 - 483z^2 + 45z - 1 = 0.$$

Here $R + s = 483/7 = 69$, and approximately $s = 45/483 = 1/10$ nearly; and hence R is a little less than 69, and we shall consequently use equation (6) in the next transformation.

Step 3. Set

$$z = \frac{69w}{w + 1}$$

to get

$$3104w^3 - 2293356w^2 + 3102w - 1 = 0.$$

Stopping at this point we obtain from equation (10) or (11)

$$s = .00135,\ 26047,\ 964,$$

whence

$$R = 738.83756,\ 49209,\ 768.$$

This gives

$$x_1 = 1 + \frac{1}{6} - \frac{1}{6 \times 69} - \frac{1}{6 \times 69 \times 738.83756,\ 49209,\ 768}$$
$$= 1.16424,\ 79384,\ 60211,\ 21305,\ 3\ \cdots.$$

As a check, Horner's method gives

$$x_1 = 1.16424,\ 79384,\ 60211,\ 21305,\ 38421,\ \cdots.$$

RECENT PUBLICATIONS.

EDITED BY W. B. CARVER, Cornell University, to whom books and communications should be sent.

REVIEWS.

Vorlesungen über die singulären Moduln und die komplexe Multiplikation der elliptischen Funktionen. By R. FUETER, Part I (Teubner's Sammlung, Band XLI, 1). Leipzig, Teubner, 1924. 142 pages. Price 7 marks.

The subject with which Professor Fueter's book deals is of central significance in the theories of elliptic functions, of quadratic number fields and of automorphic functions, and may therefore be approached from many different sides. The author has chosen to develop it on the basis of a study of the group of linear substitutions $w = (az + b)/(cz + d)$, in which a, b, c and d are integers. It is

to this subject and to the necessary preliminaries concerning quadratic fields and elliptic functions that the first part of his book is devoted.

It is well known from elementary analysis that the values of a trigonometric function for an integral multiple of θ can be expressed rationally in terms of the values of that function and of its first derivative quâ θ. The theory of elliptic functions develops the fact that, if $\phi(u)$ has the independent periods ω_1 and ω_2, *i.e.*, $\phi(u + \omega_1) = \phi(u + \omega_2) = \phi(u)$, and if k is an integer, then $\phi(ku)$ can be expressed rationally in terms of $\phi(u)$ and $\phi'(u)$. It is shown moreover that the equation determining $\phi(u/k)$ is an Abelian equation (that is, an equation whose roots are all expressible rationally in terms of any one of them), and hence solvable by radicals; and these results hold for arbitrary values of ω_1 and ω_2. When we turn to the case in which k is not rational, we have the remarkable result obtained by Abel (*Collected Works*, vol. I, p. 377) according to which an algebraic relation exists between $\phi(ku)$, $\phi(u)$ and its derivatives only in case k belongs to the field of a quadratic imaginary, and in this case only for special values of the ratio ω_2/ω_1.

The set of problems suggested by this state of affairs is commonly referred to as the problem of "complex multiplication." The moduli of the elliptic functions for which the ratio of the periods is such as to admit a "multiplication theorem" are called "singular moduli." Abel treated the problem for the case in which k belonged to the domain of $\sqrt{-3}$ or $\sqrt{-5}$, and discovered the fact that the singular moduli were expressible in terms of the ratio of the period by means of radicals. Further progress was made by Kronecker, Hermite and others, whose work brought out the strategic position of the problem. The determination of the singular moduli, the setting up and the characterization of the equations for $\phi(ku)$ have given rise to much work in algebra and the theory of numbers. So much in explanation of the title of the book under review.

To say much about its content would require a lengthier exposition of details than can well be justified in a review. For these reasons I shall limit myself to a few indications of the literature and to a brief account of the material found in this volume.

An excellent general statement of the problem is found in Weber's article in volume I of the *Encyklopaedie der math. Wissenschaften*, while a detailed study of the theory and of the many topics related to it can perhaps best be made by the aid of the third volume of Weber's *Lehrbuch der Algebra*. Although apparently conceived on a less ample basis than this classic, Professor Fueter's book may well be looked upon as continuing its tradition; it is intended to include, particularly in the second part, the important advances of recent years.

In the first chapter, on elliptic modular functions, we get an introduction to the theory of the modular group (*i.e.*, the group of linear fractional substitutions $w = (az + b)/(cz + d)$ in which a, b, c and d are integers such that $ad - bc = 1$); the elliptic modular function $j(z)$ is determined in connection with the fundamental region for the modular group. Chapter II introduces modular functions of the nth order in connection with the subgroups of the modular group and

determines the important modular equation of order n, $\Phi_n(t, j(z)) = 0$, whose roots are these modular functions.

The third chapter begins with a brief account of the fundamental properties of the ideals of a quadratic imaginary field, the notion of class number, etc. It brings the proof that the invariant $j(\omega_2/\omega_1)$ has the same value for all ideals belonging to a class; and that the class equation $H_m(t) = 0$ of the field $k(\sqrt{m})$ whose roots are the moduli $j(k_r)$, $r = 1 \cdots m$, defined for the classes $k_1 \cdots k_m$ of equivalent ideals, is an Abelian equation with respect to the field $k(\sqrt{m})$.

In chapter IV the multiplication formulæ for elliptic functions, for even and odd integral values of the multiplier, are obtained in terms of the functions $T(z)$ of Jacobi, and its derivative $T_1(z)$; these functions are related in a simple way to the Weierstrass y-function. In the last chapter it is proved that if (ω_1, ω_2) is the basis of an ideal of the field $k(\sqrt{m})$, ν an integer of this field and x_1 and x_2 integers, then $T((x_1\omega_1 + x_2\omega_2)/\nu)$ is an algebraic number. This result naturally leads then to the problem of determining the equation of lowest degree which this number satisfies and of finding the Galois group of this equation. This in turn necessitates a more detailed study of the properties of a field and the introduction of a variety of new concepts.

The book gives evidence of having been written with great care. One might wish for some variation of emphasis which would facilitate the reading and particularly the assimilation of its contents. For those who have a good knowledge of the classical theory of analytic functions and of elliptic function theory, it furnishes a very satisfactory approach to a fascinating domain of mathematics. Previous knowledge of the theory of algebraic numbers, although not presupposed, would be of material assistance. In execution the book has all the excellence which we have grown accustomed to associate with the name of Teubner.

ARNOLD DRESDEN.

The Mathematical Theory of Investment. By E. B. SKINNER. Revised edition. New York, Ginn and Company, 1924. xi + 269 pages. Price $3.40.

The first edition of Professor Skinner's book, which was published in 1913, was a pioneer American textbook in the field of mathematics of investment. It was a good book and stood alone for some years. Recently, however, a number of other textbooks of a similar nature have appeared, and it is perhaps natural that Professor Skinner should effect a revision at this time, although these newer books showed no conspicuous superiority to his original volume.

The revised edition follows the plan of the first edition, the principal changes being (1) the emphasis placed upon interpolation by the inclusion of a short chapter on that subject, and by a more extensive use of interpolation in the solution of problems involving the finding of the interest rate, and (2) the addition of a considerable amount of material on the subject of depreciation.

The feature—a desirable one—of beginning with a review of those topics of algebra which are essential in the mathematics of investment, has been retained.

It is gratifying to see included in this algebraic introduction the above-mentioned chapter on the important subject of interpolation, which for many years has unfortunately been neglected.

The chapter on graphic representation which was in the first edition has been omitted.

The equation of value and the equation of payments, formerly discussed in the chapter on interest, have been stressed by devoting to them a separate, though short, chapter. The subjects of perpetuities, capitalization, capitalized cost, continuous annuities, and annuities with fractional terms have also been placed in a separate chapter entitled "Some special types of annuities."

The author apparently neglects to state in the revised edition that the formulas for computing the premium that must be paid on a bond are equally applicable to the computation of the discount. The use of bond tables is explained, a decided improvement upon the old edition. Included in the new edition is a discussion of the determination of the rate of yield for a bond bought at a given price. The author explains three ways in which the problem may be solved: by means of bond tables, using interpolation by first and by second differences; by means of annuity tables; and by algebraic approximation. A section on the rate of interest paid by the issuer of the bond and a section on serial bonds have been added.

The chapter called "Sinking funds and depreciation" in the original edition has been broken up into three chapters in the new book: "Sinking funds," "Depreciation," and "The theory of mine valuation."

The chapter on depreciation is one of the outstanding features of the book. The author takes up the straight-line method, the sinking-fund method, the interest-on-investment method—commonly called the compound-interest method—and a modification of it, and the unit-cost method. In his discussion of the sinking-fund method, he distinguishes carefully and clearly between a *replacement* charge and a *depreciation* charge.

The discussion of the valuation of mining properties has been slightly expanded.

The chapter on building and loan associations has been somewhat simplified by removing, or putting into more compact form, certain equations.

Part III, dealing with probability and its applications to financial problems,—notably life annuities and life insurance—is practically unchanged.

New problems have been added in various places, and there are some rearrangements of material. The tables at the end of the volume are identical with those published before.

The revision, which is not at all radical, has improved the book, which seems better adapted to a fairly extensive course in the mathematics of investment than to a brief course in the subject. In addition to its value as a textbook, it is an excellent book of reference.

P. R. RIDER.

Politische Arithmetik (*Zinseszinsen-, Renten- und Anleiherechnung*). By EMIL FOERSTER. Berlin and Leipzig. Walter de Gruyter and Co., 1924. 155 pp.

This book considers the theory of simple interest, compound interest and annuities, and the applications of this theory to the usual problems of the mathematics of finance. A knowledge of college algebra is sufficient preparation for reading practically all of the book. The results and methods of development offer little that is novel, as compared with the subject matter of American books of similar content. Extensive use is made of the more refined methods of interpolation, in certain of the problems where it is more usual to rest content with the results obtained by linear interpolation. In the treatment of annuities and their applications, very little space is devoted to the cases where the payment interval of the annuity is not the same as the conversion period of the interest rate. There are numerous illustrative examples solved in the text, but there are no sets of problems for the reader to solve.

W. L. HART.

ARTICLES IN CURRENT PERIODICALS.

The lists appearing regularly in the MONTHLY of articles in current periodicals are intended to include (1) titles of papers in all mathematical journals published in the United States; (2) titles of mathematical papers and reports published by the national and state academies of science and in journals devoted to general science; (3) titles of mathematical papers by American authors published in foreign journals.

ANNALS OF MATHEMATICS, volume 26, no. 3, March, 1925: "Non-monoidal involutions which contain a web of invariant monoids" by V. Snyder, 165–172; "Definite integrals containing a parameter" by R. L. Jeffery, 173–180; "Integro-differential expressions invariant under Volterra's group of transformations" by A. D. Michal, 181–201; "New proofs of two well known theorems on quadratic forms" by J. F. Ritt, 202–204; "Some relations between compound determinants" by W. H. Metzler, 205–211; "A contribution to the theory of interpolation" by N. Wiener, 212–216; "A property of cyclotomic integers and its relation to Fermat's last theorem (second paper)" by H. S. Vandiver, 217–232; "On an infinite system of non-abelian groups of order nm^n" by W. E. Edington, 233–238; "Conformal and geodesic mapping" by A. Bramley, 239–246.

BULLETIN OF THE AMERICAN MATHEMATICAL SOCIETY, volume 31, nos. 5–6, May–June, 1925: "Life insurance as a social service and as a mathematical problem" by R. Henderson, 227–252; "Simplifications relating to Sylow's theorem" by G. A. Miller, 253–256; "Reduction of Euler's equations to a canonical form" by J. H. Taylor, 257–262; "Integro-differential equations" by I. A. Barnett, 263, 265; "A new type of double sextette closed under a binary (3, 3) correspondence" by L. D. Cummings, 266–274.

ISIS, volume 7, no. 1, November, 1924: "Drei planimetrische Aufgaben des arabischen Mathematikers Abul-jud Muhammed Ibn Al-lith" by C. Schoy, 5–8.

JOURNAL DE MATHÉMATIQUES PURES ET APPLIQUÉS, volume 4, no. 2, 1925: "On the formal modular invariants of binary forms" by W. L. G. Williams, 169–192.

SCIENTIFIC MONTHLY, volume 21, no. 2, August, 1925: "Fundamental facts in the history of mathematics" by G. A. Miller, 150–156.

TOHOKU MATH. SOCIETY, volume 25, nos. 1, 2, May, 1925: "On a series of rational functions formally analogous to Fourier's series" by W. P. Udinski, 1–23; "Determination of plane algebraic curves which are invariant under involutory Cremona transformations" by A. Emch, 63–76.

UNDERGRADUATE MATHEMATICS CLUBS.

All reports of club activities should be sent to H. J. ETTLINGER, 2910 Harris Park Ave., Austin, Texas.

CLUB ACTIVITIES.

Pi Mu Epsilon, University of Missouri, Columbia, Mo.

Pi Mu Epsilon, honorary mathematical fraternity, was founded at the University of Syracuse and incorporated under the laws of the State of New York in 1914. There are now nine chapters, the University of Missouri Chapter being the fourth.

The chapter officers for the year are:

Director: Verne T. Bickel '23.
Vice-director: W. E. Wirtel '26.
Secretary: Selma Gartman '26.
Treasurer: Richard Shewmaker '26.
Corresponding Secretary: Kathryn Wyant, Instructor in Mathematics.

The program for the year has consisted of two series of talks, one by the professors of departments related to mathematics and the other by members of the fraternity.

October 14: "The spirit of mathematics" by Professor W. D. A. Westfall.
October 28: "Mathematics of early times" by C. G. Jaeger; "Differential equations of motion" by Harlan Hibbard '25.
November 11: "Requirements for degrees at Cambridge and the mathematical tripos" by Professor L. M. Defoe.
December 9: "A problem in statistical mechanics" by Professor H. M. Reese.
December 11: Initiation.
December 29: National convention in Washington, D. C. Three delegates attended.
January 13: "Elliptic functions" by L. H. McFarlan; "Hyperbolic functions," Kathryn Wyant.
January 27: "An application of arithmetic to industry" by Professor H. Schlundt.
February 10: Report of Washington convention, by Verne Bickel.
February 29: "Historical development of laws of universal gravitation" by Professor E. S. Hayne.
March 10: "Slide rule" by R. R. Stokes '25; "Adding machine" by H. W. Wood, Jr. '25; "Planimeter" by W. E. Hoeflin '25.
March 28: Social meeting.
April 7: "The future of mathematics in high school" by M. G. Neale, Dean of School of Education.
April 28: "Laplace's equation" by J. P. Foltz '24; "Gauss's theorem" by Hilbert Moore '23; "Green's theorem" by Selma Gartman '26.
May 12: Professor G. E. Wahlin.

(Report by Miss Kathryn Wyant.)

The Pascal Circle, Trinity College, Washington, D. C.
[*1924*, 399.]

The officers for the year 1924–1925 were:

Honorary president, Professor Marie Cecilia Mangold;
President, Anne Foley, '25;
Vice-president, Helena Crowley, '25;
Secretary, Anne McLarney, '26;
Treasurer, Merie Swiney, '27.

The program for the year 1924–1925 was the following:

November 4, 1924. Report of the meetings of the past year was read by the president. It was decided to spend the time of the meetings in the study of tests in secondary schools, and in relay work in mathematical calculation by class teams.
November 18. An article on "Tests in arithmetic" was read by Miss Josephine Gillis, '24, from the report on the reorganization of mathematics in secondary schools. A discussion followed. The team work was introduced.

February 3, 1925. Margaret Harz, '27, read an article on "Algebra in the ninth grade." A discussion followed. Team work in calculation was taken up for the remainder of the period.
February 18. Ruth Eileen Lynch, '26, gave an outline of "Algebraic tests" from the report on the reorganization of mathematics in secondary schools. Relay team work.
March 3. Ruth Eileen Lynch, '26, continued her outline covering "Algebraic tests." Discussion. Relay work.
April 1. "The calculation of the date of Easter" was given by Miss Lillian Seracci, '26. Discussion.

(Report by the Secretary.)

NAPIERIAN CLUB OF DEPAUW UNIVERSITY, Greencastle, Indiana.

The Club was organized Dec. 6, 1924. On Jan. 8, 1925, the following officers were elected: President, Quinton Stone, '25; Vice-President, John Thompson, '25; Secretary, Fern Schuette, '25; Treasurer, Charlotte Stafford, '25.

The program for this year was as follows:
February 5, 1925. "Eclipse as seen from New York," Professor W. V. Brown.
February 12, 1925. "The life and work of Napier," Miss Davison, '25. "History of the decimal point," Mr. Tokey, '25.
March 19, 1925. "The squared circle," Mr. Heinzmann, '25. "Curve of pursuit," Professor W. V. Brown.
April 2, 1925. Evening spent at the observatory.
April 30, 1925. "The scale of notation," Miss Stafford, '25. "Magic squares," Mr. Arnold, Mr. Pickering, '25.

On May 14, the following officers were elected for next year:

President, James Brown, '26;
Vice-President, Ruth Bickel, '26;
Secretary, Gertrude Hendrix, '26;
Treasurer, Orin Sykes, '26.

PROBLEMS AND SOLUTIONS.

EDITED BY B. F. FINKEL, OTTO DUNKEL, AND H. L. OLSON.

Send all communications about Problems and Solutions to B. F. FINKEL, Springfield, Mo. All manuscripts should be typewritten, with double spacing and with a margin at least one inch wide on the left.

PROBLEMS FOR SOLUTION.

[N.B. Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks, or results found in readily accessible sources, will not be proposed as problems for solution in the MONTHLY. In so far as possible, however, the editors will be glad to assist members of the Association with their difficulties in the solution of such problems.]

3148. Proposed by EUGENE M. BERRY, Purdue University.

Let

$$\Delta = \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix} \neq 0$$

and X_i be the cofactor of x_i in Δ and Y_i the cofactor of y_i, etc. Prove that if

$$\begin{vmatrix} 1/x_1 & 1/x_2 & 1/x_3 \\ 1/y_1 & 1/y_2 & 1/y_3 \\ 1/z_1 & 1/z_2 & 1/z_3 \end{vmatrix} = 0$$

then

$$\begin{vmatrix} 1/X_1 & 1/X_2 & 1/X_3 \\ 1/Y_1 & 1/Y_2 & 1/Y_3 \\ 1/Z_1 & 1/Z_2 & 1/Z_3 \end{vmatrix} = 0$$

and conversely.

3149. Proposed by A. A. BENNETT, University of Texas.

Given a circle, C. Inscribe in C a simple quadrilateral whose pairs of non-adjacent sides produced meet in points P and Q respectively. Show that the circles with P and Q as centers and orthogonal to C meet in a point R, where angle PRQ is a right angle.

3150. Proposed by A. S. WIENER, Cornell University.

Prove

$$\begin{vmatrix} b^2+c^2 & 0 & -cd & be & 0 \\ 0 & a^2+d^2 & 0 & ab & ad \\ -cd & 0 & d^2+e^2 & 0 & ce \\ be & ab & 0 & b^2+e^2 & 0 \\ 0 & ad & ce & 0 & a^2+c^2 \end{vmatrix} \times \begin{vmatrix} b^2+c^2 & 0 & cd & be & 0 \\ 0 & a^2+d^2 & 0 & ab & ad \\ cd & 0 & d^2+e^2 & 0 & ce \\ be & ab & 0 & b^2+e^2 & 0 \\ 0 & ad & ce & 0 & a^2+c^2 \end{vmatrix}$$

$$= \begin{vmatrix} b^4+c^4 & 0 & -c^2d^2 & b^2e^2 & 0 \\ 0 & c^4+d^4 & 0 & a^2b^2 & a^2d^2 \\ -c^2d^2 & 0 & d^4+e^4 & 0 & c^2e^2 \\ b^2e^2 & a^2b^2 & 0 & b^4+e^4 & 0 \\ 0 & a^2d^2 & c^2e^2 & 0 & a^4+c^4 \end{vmatrix}.$$

3151. Proposed by C. C. WYLIE, University of Iowa.

"In a concave-convex lens the radii of curvature of the convex and concave surfaces are r_1 and $r_2 > r_1$. Determine the thickness and diameter of the lens so that the center of mass shall be in the concave surface." (Moulton's *Celestial Mechanics*, new edition, p. 29.)

3152. Proposed by OTTO DUNKEL, Washington University.

From two fixed points, A and B, of a given conic two chords, AC and BD, are drawn intersecting on the fixed chord, IJ, of the same conic and determining another chord, CD. Determine the envelope of the chord CD and a method for locating points on the envelope without the use of equations.

2821 [1920, 134]. Proposed by FRANK IRWIN, University of California.

The quantities $x_1, x_2, \cdots, x_n$ vary in such a way that their sum (or any other one of the elementary symmetric functions) remains constant; investigate the maxima and minima of the remaining elementary symmetric functions.

2827 [1920, 186]. Proposed by B. F. FINKEL, Drury College.

Find the equation of the envelope of the system of circles inscribed in a triangle having a given base and a given altitude.

2831 [1920, 227]. Proposed by B. J. BROWN, Kansas City, Mo.

Given that the series $\sum_0^\infty a_n y^n$ is absolutely convergent when $|y| < 1$, prove that the series $\sum_0^\infty a_n(2x\cos\theta - x^2)^n$ may be arranged in powers of x provided that $|x| < 2/5$.

Prove that if x lies between 1 and 2 the series $\sum_0^\infty (2x - x^2)^n$ is convergent and consists only of positive terms, but that the series obtained by arranging it in powers of x diverges.

2841 [1920, 274]. Proposed by WILLIAM HOOVER, Columbus, Ohio.

The square number,

$$9\frac{49}{64} = 9 + \frac{49}{64} = 3^2 + \frac{7^2}{8^2}$$

is of the type form

$$k^2 + \frac{(2k+1)^2}{(2k+2)^2};$$

how may the *forms* of the terms of the fractional part be determined *deductively*?

Generally, required that

$$k^2 + \frac{\{\varphi_1(k)\}^2}{\{\varphi_2(k)\}^2}$$

be a perfect square, show how $\varphi_1(k)$ and $\varphi_2(k)$ may be found.

SOLUTIONS.

2706 [1918, 216]. Proposed by H. F. MACNEISH, New York City.

Through a given point draw a straight line cutting a given straight line and a given circle, such that the part of the line between the given point and the given line may be equal to the part within the circle.

SOLUTION BY LOUIS WEISNER, University of Rochester.

The following analysis shows that in general the construction is impossible with straight edge and compass alone.

Take the given point O as origin, and as x-axis the line joining O and the center C of the circle. Let the equation of the given line be $y = mx + b$, and the equation of the circle

$$(x - c)^2 + y^2 = a^2.$$

The equation of the required line is of the form $y = kx$, where k is to be determined; let it cut the given line in A and the circle in $B(x_1, y_1)$ and $D(x_2, y_2)$. We readily find that the coördinates of A are $\left(\frac{b}{k-m}, \frac{kb}{k-m}\right)$ and hence $\overline{OA}^2 = \frac{(1+k^2)b^2}{(k-m)^2}$.

Substituting $y = kx$ in the equation of the circle, we find that x_1 and x_2 satisfy the equation,

$$(1 + k^2)x^2 - 2cx + c^2 - a^2 = 0.$$

Without solving this quadratic equation, we find that

$$(x_1 - x_2)^2 = \frac{4[(a^2 - c^2)k^2 + a^2]}{(1 + k^2)^2}.$$

Now, as $y_1 = kx_1$ and $y_2 = kx_2$,

$$\overline{BD}^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2 = (1 + k^2)(x_1 - x_2)^2 = \frac{4[(a^2 - c^2)k^2 + a^2]}{1 + k^2}.$$

Since $\overline{OA} = \overline{BD}$, we have the following equation to solve for k:

$$4(k - m)^2([a^2 - c^2]k^2 + a^2) = b^2(1 + k^2)^2.$$

Remembering that a, b, c, m are independent given quantities, it is quite clear that, when this equation is simplified, the terms being arranged according to powers of k, we shall have a *general* equation of the fourth degree. Now the solution of a general biquadratic equation always involves an irreducible cube root. The construction is therefore, in general, impossible with straight edge and compass alone.[1]

3093 [1924, 353; 1925, 267]. Proposed by FRANK MORLEY, Johns Hopkins University.

A solution by the PROPOSER was overlooked when the solution of this problem was printed (*1925*, 267). EDITOR.

[1] *Cf.* Miller, Blichfeldt and Dickson, *Finite Groups*, p. 322.

3114 [1925, 94]. Proposed by B. R. HEADSTROM, Boston, Mass.

What is the per cent. error in the following supposed method of trisecting an angle? Given the angle ABC with its vertex at B, lay off on its sides the equal lengths BD and BE and draw the line DE. With this latter length as diameter draw a semi-circle on the side opposite to B; also draw arcs of circles with D and E as centers and radii equal to that of the semi-circle, cutting the latter in M and N, respectively. Then the lines BM and BN trisect the given angle.

SOLUTION BY B. F. FINKEL, Springfield, Mo., and OTTO DUNKEL, Washington University.

Let O be the middle point of DE, and draw the straight line BO producing it until it cuts NM in P. Since $\angle PON = 30°$ it will be convenient to take $OE = 2$. Let the angle to be trisected $\angle DBE = 2\theta$ and set $\angle OBN = \alpha$. Then $\cot \alpha = BP/PN = \sqrt{3} + 2\cot\theta$. The error in taking $\angle NBE$ as $2\theta/3$ is $\epsilon = 2\theta/3 - (\theta - \alpha) = \alpha - \theta/3$. Hence the per cent. of error is found from

$$\frac{\epsilon}{2\theta} = \frac{\cot^{-1}[\sqrt{3} + 2\cot\theta]}{2\theta} - \frac{1}{6},$$

$$= \frac{1}{2\theta}\tan^{-1}\left[\frac{\tan\theta}{2 + \sqrt{3}\tan\theta}\right] - \frac{1}{6}.$$

If θ is very small, then, approximately,

$$\frac{\epsilon}{2\theta} = \frac{1}{4} - \frac{1}{6} = \frac{1}{12}.$$

Setting $\epsilon = 0$, it is found that $\tan\theta/3 = 0,\ \pm 2 - \sqrt{3},\ 1/\sqrt{3}$; hence the construction is exact for the angles $2\theta = 0, 90°, 180°, \cdots$.

The maximum error may be found from

$$\frac{3}{2}[\tan^2\theta + (\sqrt{3}\tan\theta + 2)^2]\frac{d\epsilon}{d\theta} = (\tan\theta - \sqrt{3} + \sqrt{2})(\tan\theta - \sqrt{3} - \sqrt{2}).$$

Hence $d\epsilon/d\theta$ is zero for

$$\theta_1 = 17° 37' 55.9'', \qquad \theta_2 = 72° 22' 4.1'',$$
$$\epsilon_1 = 1° 13' 33.7'', \qquad \epsilon_2 = -\epsilon_1.$$

Thus as the angle 2θ increases from 0 to $2\theta_1$ the error increases from 0 to ϵ_1, and as the angle continues to increase to 90° the error decreases from ϵ_1 to 0. As the angle increases from 90° to $180° - 2\theta_1$ the error decreases from 0 to $-\epsilon_1$, and after this point the error increases until 2θ is 180° when the error is again zero.

For $2\theta = 45°$, $\epsilon = 1° 10'$; for $2\theta = 20°$, $\epsilon = 1° 2.42'$. A better approximation is given in the problem and its solution **2972 [1925, 95]**.

NOTES AND NEWS.

Readers are invited to contribute to the general interest of this department by sending items to H. W. KUHN, Ohio State University, Columbus, Ohio.

Dr. IRVING LANGMUIR, of the General Electric Company, Schenectady, has been awarded the Cannizzaro prize of the Reale Accademia dei Lincei.

Trinity College, Dublin, has conferred the honorary degree of doctor of science on Professor R. A. MILLIKAN, of the California Institute of Technology.

Miss ALICE A. GRANT has been appointed instructor of mathematics at State College of Washington.

Mr. W. H. McEwen has been appointed instructor of mathematics at Regina College, Regina, Canada.

Mr. E. L. Mickelson has been appointed instructor of mathematics at the University of Wisconsin.

Miss Marcia Latham, of Hunter College, died May 9, 1925. Her translation of Descartes' *Geometry*, prepared in collaboration with Professor D. E. Smith, recently appeared through the Open Court.

The new Distinguished Service professorships at the University of Chicago are a part of the $17,500,000 development program that is now under way. Each one carries a salary of $10,000 and is subject to assignment, not to any particular department, but to whatever individual is deemed most worthy on the basis of his record. The selection of Professor Michelson as the holder of the first of these professorships was determined by selective ballot of the members of the University Senate consisting of all full professors. By oversight this appointment was called a "fellowship" instead of a professorship in the August–September issue of the Monthly.

The ninth Summer meeting of the Association at Ithaca was unusually successful both in the large number attending and in the spirit of the occasion. No little interest was added on account of the Colloquium lectures of the American Mathematical Society which were held during the week. The lectures were given by Professor L. P. Eisenhart on "The New Differential Geometry" and Professor Dunham Jackson on "The Theory of Approximation." The attendance was considerably larger than on any previous colloquium. The report of these lectures and of the meetings of the Society will appear in the Bulletin of the Society. The report of the Association meetings will appear in full in an early issue of the Monthly. Numerous matters of interest and importance occupied the attention of the Trustees during two long sessions.

SECOND ANNUAL MEETING OF NEBRASKA SECTION.

The second annual meeting of the Nebraska Section was held at the University of Nebraska, at Lincoln, on May 9, 1925, in affiliation with the Nebraska Academy of Science.

The following papers were presented:

(1) "Factorization of large numbers" by Mr. J. E. OPP, University of Nebraska.

(2) "Constructions of the triangle" by Mr. E. B. OGDEN, University of Nebraska.

(3) "A class of summable trigonometric series" by Mr. G. D. NICHOLS, University of Nebraska.

(4) "Symbolic forms for derivatives of products and determinants" by Professor W. C. BRENKE, University of Nebraska.

(5) "Are the standards and requirements in mathematics being lowered—

(*a*) in the high school?" by Miss ELLEN H. FRANKISH, North High School, Omaha.

(*b*) in the first year courses in college?" by Professor J. M. HOWIE, Nebraska Wesleyan University.

Abstracts of the papers:

(1) Mr. Opp dealt with the factorization of large numbers by the use of a quadratic irrationality and the construction of a table of linear forms of numbers which must contain one of the required factors.

(2) Mr. Ogden gave an outline of methods used in solving, by ruler and compass construction, about one thousand plane triangles. About two hundred and fifty of these problems are either solved or suggested in various texts on geometry; the others are original.

(3) Mr. Nichols established the following theorem on the sum of a non-convergent series, using the Cesaro mean value definition of "sum." "The series $\sum n^p \sin nx$ is summable of order k, if $k - 1 < p < k$."

(4) Professor Brenke showed how the formula

$$\frac{\partial^{p+q}}{\partial x^p \partial y^q}(uvw) = (u + v + w)_x{}^p (u + v + w)_y{}^q$$

can be regarded as giving the indicated derivative of the product uvw, or of the determinant whose elements by columns are u_i, v_i, w_i, $i = 1, 2, 3$, with extension to the general case.

(5*a*) Miss Frankish expressed the opinion that the standards and requirements in mathematics have not been lowered, during the last ten years, in the Omaha Central High School, which is a large school with about 80 per cent. of its graduates going to college. Increasing difficulty arises in keeping up to college entrance